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Utility Maximization with Consumption Habit Formation in Incomplete Markets

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Utility Maximization with Consumption Habit Formation in Incomplete Markets

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Dedicated to my family.

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Utility Maximization with Consumption Habit Formation in Incomplete Markets

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This dissertation studies a class of path-dependent stochastic control problems with applications to Finance. In particular, we solve the open problem of the continuous time expected utility maximization with addictive consumption habit formation in incomplete markets under two independent scenarios.

In the first project, we study the continuous time utility optimization problem with consumption habit formation in general incomplete semimartingale financial markets. Introducing the set of auxiliary state processes and the modified dual space, we embed our original problem into an abstract time-separable utility maximization problem with a shadow random endowment on the product space $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$. We establish existence and uniqueness of the optimal solution using convex duality by defining the primal value function as depending on two variables, i.e., the initial wealth and the initial standard of living. We also provide market independent sufficient conditions

both on the stochastic discounting processes of the habit formation process and on the utility function for the well-posedness of our original optimization problem. Under the same assumptions, we can carefully modify the classical proofs in the approach of convex duality analysis when the auxiliary dual process is not necessarily integrable.

In the second project, we examine an example of the optimal investment and consumption problem with both habit-formation and partial observations in incomplete markets driven by Itô processes. The individual investor develops addictive consumption habits gradually while only observing the market stock prices but not the instantaneous rates of return, which follow an Ornstein-Uhlenbeck process. Applying the Kalman-Bucy filtering theorem and Dynamic Programming arguments, we solve the associated Hamilton-Jacobi-Bellman(HJB) equation fully explicitly for this path dependent stochastic control problem in the case of power utility preferences. We provide the optimal investment and consumption policy in explicit feedback form using rigorous verification arguments.

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Chapter 1

Introduction

1.1 Economic Motivation for Habit Formation Preferences

The question of how to optimally choose an investment and consumption policy in financial markets is a central problem in Mathematical Finance and Financial Economics. One critical step is the appropriate way to measure the investors' happiness level and risk appetite. The traditional way to model such preferences is by defining a concave and non-decreasing function on the terminal wealth state or on the intermediate consumption rate process. Such a preference is also referred as the von Neumann-Morgenstern utility function. It has a lot of satisfactory features that can capture some important economic phenomena and provide a quantitative indicator of investor's attitude to risk. The von Neumann-Morgenstern utility preference has been widely used and analyzed for a long time.

However, during the past decades, the assumption of time-additivity of von Neumann-Morgenstern preferences on consumption plans has been challenged due to its lack of consistency with many observed empirical evidences. For instance, the celebrated magnitude of the equity premium (*Mehra and Prescott* [54]) can not be reconciled with the time separable preferences

$\mathbb{E}[\int_0^T U(t, c_t)dt]$ when the instantaneous utility function U is only derived from the consumption rate. More precisely, compared with the theoretical data by the consumption-based capital asset pricing model with time separable power utility in the complete market, the excess returns of stocks over the risk-less assets appear to be considerably high. By postulating that the representative agent has the time separable utility preference in the equilibrium model, *Mehra and Prescott* [54] show that the mismatch of high returns on risky assets can only be explained by assuming consumers will feel painful even if the intertemporal consumption fluctuates slightly. In other words, one must assume that consumers are implausibly risk averse.

As an alternative modeling tool, habit formation has attracted a lot of attention and has been actively investigated in recent years. This new way to compare consumption streams is defined by $\mathbb{E}[\int_0^T U(t, c_t, Z_t)dt]$, where the accumulative process Z_t, called the standard of living or habit formation process, describes the consumption history impact. Moreover, we assume the instantaneous function U is increasing c, decreasing in Z and concave in both processes. In particular, the accumulative process $Z_\cdot \triangleq Z(\cdot; c)$ is defined in the following way:

$$dZ_t = (\delta_t c_t - \alpha_t Z_t)dt,$$

$$Z_0 = z,$$

where the stochastic discounting factors α_t and δ_t are generally assumed to be nonnegative optional processes and the given real number $z \geq 0$ is called the *"initial habit"* or *"initial standard of living"* .

From its definition, we can see that the habit formation preference possesses the potential to answer the above equity premium puzzle. Compared to the utility form consumption itself, the change in consumption plans displayed much larger relative variance in the habit-adjusted consumption. In other words, a small drop in consumption may cause a large fluctuation in consumption net of the subsistence level due to the standard of living constraint and hence can possibly explain sizable excess returns on risky assets in equilibrium models even for moderate values of the degree of risk aversion. Based on this, there is a vast literature that recommends the habit formation preference as the new economic paradigm which can resolve the equity premium puzzle as well as many other empirical observations, we refer the readers to, for instance, *Constantinides* [15], *Samuelson* [68] and *Campbell and Cochrane* [14].

At the intuitive level, the other remarkable feature of the habit formation preference is its reflection of consumers' rationality from the psychological perspective. In contrast to the traditional time additive utilities, the concept of habit formation characterizes the non-negligible effect of past consumption patterns on current and future economic decisions. Consumption behaviors in daily life often are repetitive and performed in customary places, leading consumers to develop habits. And if the investor has been living a steady life style for a reasonable long time, even during the economic recession period, he/she may still be willing to sacrifice savings in order to protect the living standards. This reveals that high consumption history will generically lift up the investor's desired consumption plan for the future. And back to the precise definition of habit formation preference, it specifies that the utility of consumption at time t depends also negatively on the history of consumption up to time t, which in particular means an increase in consumption today increases current utility but depresses all future utilities through the induced increase in future standards of living.

The study of habit formation in modern economics dates back to *Hicks* [33] in 1965 and *Ryder and Heal* [67] in 1973. More recently, the utility maximization problem with consumption habits in continuous time has been studied by *Constantinides* [15] to explain the equity premium puzzle. In complete Itô processes markets, *Detemple and Zapatero* [24] and [25] employ martingale methods to study the general nonlinear habit formation utility optimization problem $\mathbb{E}[\int_0^T U(t, c_t, Z_t)dt]$ and establish some recursive stochastic differential equations for the consumption rate process c_t. They derive a closed form solution for the optimal consumption under preferences of the type $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$ when $U : [0, \infty) \times (0, \infty) \to \mathbb{R}$, i.e., the habit is assumed to be *linear* and *addictive*. Later, *Schroder and Skiadas* [71] make an insightful observation that to solve the optimal portfolio selection with utilities incorporating linear habit formation $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$ in the complete market is equivalent to solving the time separable utility maximization $\mathbb{E}[\int_0^T U(t, c'_t)dt]$ in the isomorphic complete market without habit formation. The isomorphism is given by the relation that the optimal policies $c'^*_t = c^*_t - Z^*_t$ holds true. They also give the construction of the isomorphic market based on the original market under some appropriate assumptions. *Munk* [58] applies the Market Isomorphism result in the complete market model with mean reverting drift process and stochastic interests rates process, and provides closed form optimal strategies in several special cases. *Detemple and Karatzas* [23] further consider the linear non-addictive habits $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$, where instead they define $U : [0, \infty) \times (-\infty, \infty) \to \mathbb{R}$. Their consumption c_t is required to be non-negative but is allowed to fall below the *"the standard of living"* index Z_t that aggregates past consumption. They provide a constructive proof for the existence of an optimal consumption, however, the market completeness is still a key assumption for their analysis. *Egglezos and Karatzas* [26] exploit the interplay between stochastic partial differential equations and the utility maximization with linear addictive habit formation by taking advantage of the first order condition in the non-Markovian complete market, therefore obtaining stochastic feedback formulae for the optimal portfolio and consumption policies. Although significant progress has been made in the complete market setting, it is still a well-known open problem to investigate the existence of optimal consumption policy for the habit-forming investor under utility maximization when the financial market is incomplete, which consists one of the primary motivations of our present work.

1.2 Convex Duality in Utility Maximization Problems

We present in this section an overview of the convex duality in utility maximization problem which will be the main tool for our analysis in Chapter 2 in the framework of incomplete semimartingale financial markets.

The single agent optimal portfolio and consumption problem in stochastic framework dates back to *Merton* [55], [56]. Assuming the market assets follow Markovian Itô processes, *Merton* derives and solves the Hamilton-Jacobi-Bellman (HJB) equation for the utility function of the HARA case by employing the Dynamic Programming Principle, and accordingly obtains the so called feedback form of the optimal investment and consumption policies. His pioneering work opened the door for the assistance of nonlinear PDE analysis to this type of stochastic control problems, and stimulated a recent upsurge of exciting academic research in both the theoretical PDE field and the stochastic control methods associated to financial applications.

If the setup is that of general non-Markovian diffusion or semimartingale models, dynamic programming arguments based on HJB equations can not be applied. As a powerful alternative approach, convex duality in stochastic control was first introduced by *Bismut* [8], which later demonstrated its merit of feasibility of the analysis in general market models and for general utility preferences. Starting from 1980s, a series of papers applying this convex analysis and martingale methods for the optimal portfolio and consumption policy in complete markets have been published, i.e., *Pliska* [62], *Karatzas, Lehoczky and Shreve* [39] and *Cox and Huang* [16] [17]. In essence, they successfully represent the optimal terminal wealth (consumption process) as the inverse marginal utility of the unique local martingale measure (density process). However, in incomplete markets, the fact that these equivalent local martingale measures become infinitely many invalidates the direct relation between the original optimization problem and the dual variational problem. On the other hand, it is also unclear whether the dual problem itself can achieve its optimal value over the set of equivalent local martingale measures. For the purpose to obtain the analogous conjugate duality relations, early work, as *He and Pearson* [32], and by *Karatzas, Lehoczky, Shreve and Xu* [40], propose to complete the original markets by some fictitious stocks and solve the dual optimization over a set of carefully defined parameterized local martingales with the investment constrained in a least favorable manner.

Kramkov and Schachermayer [48], [49] are acknowledged for their pioneering work to treat the problem of optimal investment in the context of general incomplete semimartingale financial markets. By demonstrating the set of the local martingale deflators in the previous work is too small for the existence of the dual problem, they enlarge it to a suitable chosen set $\mathcal{Y}$ of supermartingales whose solid hull verifies to be the smallest closed, convex and solid set containing the previous set of equivalent local martingale measures in $\mathbb{L}^0_+$ endowed with the topology of convergence in probability. Moreover, they give an affirmative answer to the theoretical existence of optimal solutions in both the primal problem and dual problem using the minimal sufficient conditions on the asymptotic behavior of the utility function, which is called the

Reasonable Asymptotic Elasticity condition. This is now the customary assumption for utility maximization problems in the semimartingale framework. One critical technique that their proofs rely heavily on is the celebrated Bipolar theorem on $\mathbb{L}^0_+$ proved by *Brannath and Schachermayer* [11], which extends the classical result in functional analysis for locally convex vector spaces to the space $\mathbb{L}^0_+$. They identify the bipolar of a subset $\mathcal{C}$ of $\mathbb{L}^0_+$ as the smallest convex, closed in probability and solid set containing $\mathcal{C}$. Later, the theorem was generalized for sets of stochastic processes, and *Žitković* [76] derived the filtered version of the bipolar theorem motivated by the problem of optimal intertemporal consumption choice for an investor in incomplete semimartingale markets.

One interesting variation problem arises later when an exogenous random endowment is taken into account. *Cvitanic, Schachermayer and Wang* [18] observe that the previous set $\mathcal{Y}$ of supermartingale deflators can no longer host the optimal solution for the associate dual problem, and they suggest to consider the space $\mathcal{D}$ of bounded finitely additive measures, which is the $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$ weak-$*$ closure of the set $\mathcal{M}$. Specifically, there exists an additional singular part associated to the equivalent local martingale measure which can not be ignored by the presence of the random endowment. Later, the model incorporating with the intermediate consumption rate process and stochastic clock are examined by *Karatzas and Žitković* [43] and *Žitković* [77], where the optional decomposition theorem for supermartingales by *Kramkov* [47] and *Föllmer and Kramkov* [28] plays a very important role. Fortunately, the previous work build the relationship between the primal optimizer only to the unique regular part of a bounded finitely additive measure, which is the optimal solution of the dual problem. This observation prompts the use of this enlarged dual space $\mathcal{D}$ in spite of its abstract definition. Moreover, *Karatzas and Žitković* [43] and *Žitković* [77] show that optimal dual process carefully defined by the regular part of the finitely additive measure turns out to be a discounted supermartingale process, which retains the benefit that it can be approximated by some equivalent local martingale measure densities.

On the other hand, in order to to avoid the extension of the dual space to finitely additive measures, *Hugonnier and Kramkov* [35] add more dimensions to the primal value function to deal with the random endowment. To be precise, they define their value function both on the initial wealth x and the vector q which is the number of units of each contingent claim. They can hide the sophisticated singular part of the finitely additive measures into the definition of the effective domain of values x and q, and construct the appropriately modified dual space for the convex duality to hold true.

The convex duality approach now plays a very important role in the treatment of general utility maximization problems in the framework of incomplete markets under different types of constraints. The feasible applications of this approach in many different models and problems have brought an increasing popularity and many scholars are currently working for various possible extensions in different directions. To list a very small subset of the very recent literature in optimal investment and consumption problems, we refer to

Bouchard and Pham [10], *Biagini* [6], *Biagini and Frittelli* [7], *Kauppila* [44] and *Larsen and Žitković* [53].

1.3 Incomplete Information and Kalman-Bucy Filtering

We briefly present here the linear stochastic filtering and Kalman-Bucy filtering theorem which will be used later in Chapter 3 to deal with the utility maximization problem with consumption habit formation and partial observations.

Consider a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a background filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$. All processes are assumed to be $\mathbb{F}$ progressively measurable and are one dimensional for simplicity. The stochastic signal process $R_t:\ \Omega \longrightarrow \mathbb{R}$ is given by the SDE

$$dR_t = A(t, R_t)dt + C(t, R_t)dB_t, \quad 0 \leq t \leq T,$$

where B denotes a $\mathbb{F}$-Brownian Motion. We assume the signal process R_t is not directly observable.

And an observation process H_t is given by

$$dH_t = D(t, R_t)dt + E(t, R_t)dW_t, \quad 0 \leq t \leq T,$$

where, W is another $\mathbb{F}$-Brownian motion, correlated with B with the coefficient $\rho\ \in [-1, 1]$.

The stochastic filtering problem is to find the best estimate $\hat{R}_t$ based on the observation of H_t: which means to compute the conditional distribution of the signal process $\hat{R}_t$, given the observation filtration $\hat{\mathfrak{F}}_t = \sigma(H_s : 0 \le s \le t)$:

$$\mathbb{P}\Big[R_t \in A \Big| \hat{\mathfrak{F}}_t\Big], \quad 0 \le t \le T, \ \text{ for } \ A \in \hat{\mathfrak{F}}_T.$$

Equivalently, this amounts to compute the conditional expectation:

$$E\Big[f(R_t)\Big|\hat{\mathfrak{F}}_t\Big], \quad 0 \le t \le T,$$

for a suitable class of test functions $f : \mathbb{R} \to \mathbb{R}$.

As a special case, we now assume the one dimensional signal state process R_t and observation process H_t satisfy the following linear SDEs:

$$\begin{aligned} dR_t &= A(t)R_t dt + C(t)dB_t \\ dH_t &= D(t)R_t dt + E(t)dW_t, \quad 0 \le t \le T, \end{aligned} \tag{1.3.1}$$

where, $A(t),\ C(t) \neq 0,\ D(t),\ E(t) \neq 0$ are deterministic functions satisfying the integrability condition:

$$\int_0^T \Big[|A(t)| + |D(t)| + C^2(t) + E^2(t)\Big] dt < +\infty.$$

Suppose the $\mathfrak{F}_0$-measurable random variable R_0 is Gaussian distributed with $R_0 \sim N(\mu, \theta)$, which is independent of Brownian motions $(B_t)_{0 \le t \le T}$ and $(W_t)_{0 \le t \le T}$. And $H_0 = s$, where s is a constant.

One can see that (R_t, H_t) is a Gaussian vector process, so the conditional distribution of R_t given $\hat{\mathfrak{F}}_t = \sigma(H_s : 0 \le s \le t)$ is also Gaussian, see the complete description in section 5.5.6 of *Karatzas and Shreve* [41]. Therefore the entire conditional distribution is determined by the conditional mean $\hat{R}_t$

and conditional variance $\hat{\Omega}_t$:

$$\hat{R}_t = \mathbb{E}\Big[R_t\Big|\hat{\mathfrak{F}}_t\Big],$$
$$\hat{\Omega}_t = \mathbb{E}\Big[(R_t - \hat{R}_t)^2\Big|\hat{F}_t\Big],$$

with initial values

$$\hat{R}_0 = \mathbb{E}\Big[R_0\Big|\hat{\mathfrak{F}}_0\Big] = \mathbb{E}\Big[R_0\Big] = \mu$$
$$\hat{\Omega}_0 = \mathbb{E}\Big[(R_0 - \hat{R}_0)^2\Big|\hat{\mathfrak{F}}_0\Big] = \mathbb{E}\Big[(R_0 - \mu)^2\Big] = \mathrm{Var}\Big[R_0\Big] = \theta.$$

The linear filtering problem amounts to computing the $\hat{R}_t$ and $\hat{\Omega}_t$ given their initial values. The celebrated Kalman-Bucy filter theorem gives us the algorithm to compute these estimations.

Theorem 1.3.1 (One-dimensional Kalman-Bucy Filtering)**.**

For the one-dimensional signal state process R_t and observation process H_t following the linear SDEs (1.3.1)*, the conditional expectation $\hat{R}_t = \mathbb{E}\Big[R_t\Big|\hat{\mathfrak{F}}_t\Big]$ satisfies the SDE:*

$$d\hat{R}_t = A(t)\hat{R}_t dt + \Big[\frac{D(t)}{E(t)}\hat{\Omega}_t + C(t)\rho\Big]d\hat{W}_t, \quad 0 \le t \le T,$$

*with $\hat{R}_0 = \mu$, where $(\hat{W}_t)_{0\le t\le T}$ is called the **Innovation Process**, and definded as:*

$$d\hat{W}_t = dW_t + \frac{D(t)}{E(t)}\Big(R_t - \hat{R}_t\Big)dt, \quad 0 \le t \le T, \quad \hat{W}_0 = 0.$$

Moreover, the innovation process $\hat{W}$ is an $(\hat{\mathfrak{F}}_t)_{0\le t\le T}$ adapted Brownian motion.

The conditional variance $\hat{\Omega}_t = Var\Big[R_t\Big|\hat{\mathfrak{F}}_t\Big]$ satisfies the deterministic Riccati equation:

$$\frac{d\hat{\Omega}_t}{dt} = -\frac{D^2(t)}{E^2(t)}\hat{\Omega}_t^2 + 2\Big[A(t) - \frac{C(t)D(t)\rho}{E(t)}\Big]\hat{\Omega}_t + C^2(t)(1-\rho^2), \quad 0 \le t \le T,$$

with the initial condition $\hat{\Omega}_0 = \theta$.

For the sake of completeness, the proof of Kalman-Bucy Filtering Theorem is presented in the Appendix B.

1.4 Outline of the Dissertation

In this dissertation, we only focus on a special family of consumption habit formation preferences, namely the *addictive linear* habit formation. The preferences are defined in the sense that the instantaneous utility function depends on the difference of consumption rate process and the standard of living process, i.e., $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$, where the domain of instantaneous utility function is given by $(0, \infty)$ and for each $t \in [0, T]$. We extend the definition of U by $U(t, x) = -\infty$ for all $x < 0$, which is equivalent to the lower bound constraint on consumption rates by requiring that the consumption c_t shall never fall below the current *"standard of living"* Z_t. As we pointed out in the survey of Financial Economics literature, this type of consumption habit formation preference became increasingly popular among academic fields, in virtue of its capability to capture the influence of addictive consumption patterns on current economic decisions together with its mathematical tractable structures compare to general *nonlinear* habit formation preferences.

The rest of this dissertation consists of three chapters. Chapter 2 and Chapter 3 are two independent projects studying optimal investment and consumption with habit formation in incomplete markets in two separate scenarios. Possible future extensions of our current work are presented in the final Chapter 4. The summarized description of the contents in each chapter is listed as follows:

Chapter 2 treats a well-known open problem in Mathematical Finance and Financial Economics, i.e., the utility maximization problem with consumption habit formation in general incomplete semimartingale markets. Our ultimate goal is to obtain the existence and uniqueness of the optimal consumption plan with the assistance of convex duality. However, the straight-forward application is hindered by the path dependent structure of the habit formation preference. Difficulties emerge in both the primal optimization problem and its dual problem. Indeed, on the one hand, the admissible space for consumption process c_t loses its *solidity* property due to the addictive habit formation constraint. This prevents the direct derivation of the primal optimizer by a standard argument. On the other hand, the time non-separability implies the path dependence of its dual problem which also makes it difficult to show the lower semi-continuity with respect to the converging dual processes. The existence of optimal solution of the formally defined dual problem itself becomes complicated to tackle.

Generalizing the idea of Market Isomorphism developed by *Schroder and Skiadas* [71], we introduce the closely related auxiliary process $\tilde{c}_t = c_t - Z_t$ for each financeable consumption rate process c_t, and define the utility maximization over the set of auxiliary processes to overcome this intrinsic path-dependent complexity. By introducing the auxiliary dual process Γ_t defined via the equivalent local martingale measure density process Y_t, a key equality easily derived by Fubini-Tonelli's theorem, enables us to shift the budget constraint from consumption rate process c_t with respect to the process Y_t to the auxiliary process $\tilde{c}_t$ with respect to the auxiliary process Γ_t. We show that the set of $\tilde{c}_t$ preserves the closedness and convexity properties. Moreover, it is also solid, which provides us the natural way to treat it as the primal process. However, the price we need to pay is the extra integral of the exogenous term $\tilde{w}_t$ with respect to the auxiliary process Γ_t, where $\tilde{w}_t$ is fully determined by the stochastic discounting processes δ_t and α_t. To exclude the easier case, we make the preliminary assumption that the extra term appeared in the characterization of the set for $\tilde{c}_t$ is not replicable. We choose to add one more dimension in the definition of the primal value function and treat the value function as depending on both the initial wealth x and initial habit z, and formally embed our original problem into the framework of *Hugonnier and Kramkov* [35]. By defining the properly modified dual domain and enlarge the effective domain for values of x and z, we are eventually able to embed our optimization problem into an abstract time separable utility maximization problem with the shadow random endowment, and prove the existence and uniqueness of the optimal consumption strategies as well as the conjugate duality relations between values functions.

Another unique and interesting feature of our problem that should be highlighted is the relaxation of the integrability of the auxiliary dual processes, which is a consequence of the unboundedness of the stochastic discounting processes δ_t and α_t. This subtle issue of integrability requires us to make some additional market independent assumptions for the well posedness of the utility maximization problem and to modify the analysis of the convex duality. As it is well known, the classical proofs rely heavily on the fact that the dual space is a subset of $\mathbb{L}^1$. In particular, we need to control the asymptotic behavior of the utility function when x goes to 0 as well as to impose the assumption that one integral involving δ_t and α_t has a lower bound. In the proofs of our main results, we successfully extend the convex duality to the product spaces $\mathbb{L}^0(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$, and all the desired consequences and properties are retained.

As an independent project, Chapter 3 studies a similar utility maximization problem, however, in an incomplete Itô processes market under incomplete information. Our consumption habit formation preference is again defined via $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$ where the instantaneous utility function is assumed to be the power utility function $U(t,x) = \frac{x^p}{p}$ for $p < 1$ and $p \neq 0$. The optimal control aims to maximize the investor's utility over consumption with habit formation as well as the terminal wealth. In terms of partial observations, we restrict the investor's access to the whole market information, and he/she can only observe the stock price published to the public, but can not observe the Brownian motion driving the stock process and instantaneous drift process which satisfies a mean reverting Ornstein Uhlenbeck SDE. Therefore, the investor ought to choose his optimal portfolio and consumption policies adapted to the partial observations filtration $\mathcal{F}^S$ which is generated by the stock price process S_t. By applying the Kalman-Bucy filtering theorem, we can rewrite all the market state processes driven by the innovation process which itself is a Brownian motion under the partial filtration $\mathcal{F}^S$. To avoid the complicated direct proof of Dynamic Programming Principle, we choose the analytic approach to formally derive the Hamilton-Jacobi-Bellman (HJB) equation of function $V(t, x, z, \eta)$ by using Itô Lemma and show the smooth solution of the HJB equation equals the value function using rigorous verification arguments. To this end, the first step is to reduce the dimension of the target function by noticing the conditional variance is a deterministic function. Later, by the homogeneity of the power utility function, we guess the solution as an priori decoupled form $V(t, x, z, \eta) = \frac{(x-W(t)z)^p}{p} M(t, \eta)$, and solving the HJB equation is simplified to solving a nonlinear PDE for the function $M(t, \eta)$. As the third step, by the well-known power transform, we can simplify the nonlinear PDE of $M(t, \eta)$ to the linear PDE for the function $N(t, \eta)$ where we have the relation $M(t, \eta) = N(t, \eta)^{1-p}$.

Due to the special structure of the linear PDE for $N(t, \eta)$, the solution can be further deduced in a closed form, and the algorithm simplifies to solving three ODEs with time dependent parameters. With the aid of the elegant observation made by *Brendle* [12] for the optimal wealth optimization problem, we can further transform our ODEs with time dependent parameters to a system of ODEs with constant parameters. More importantly, the fully explicit solutions of the latter family of ODEs can be gained in four different cases depending on the values of the market parameters. Eventually, we prove the verification lemma and the optimal portfolio and consumption strategies are extracted in feedback form. Some conditions on the market parameters and the risk aversion constant p are demanded for the sake of technical proofs of some uniform integrability results.

We present the future research in Chapter 4. We discuss several possible directions to extend our current work including the non-addictive habit formation and nonlinear consumption habit formation as well as nonlinear stochastic filtering problems. Some difficulties to associated problems are also revealed respectively in the last chapter.

Chapter 2

Utility Maximization with Consumption Habit Formation in Incomplete Semimartingale Markets

2.1 Introduction

In this chapter, we consider the utility maximization problem with consumption habit formation in the framework of general incomplete semimartingale markets. As already mentioned, it is, to the best of our knowledge, an important open problem. In the words of *Egglezos and Karatzas* [26], "The existence of an optimal portfolio/consumption pair in an incomplete market (that is, when the number of stocks is strictly smaller than the dimension of the driving Brownian motion), is an open question. ..., and new methodologies are needed to handle the problem."

In this chapter, we are interested in the most general framework and therefore allow all the discounting factors of habit formation index to be unbounded nonnegative optional processes in the given probabilistic setting.

Typically, if we choose the conventional enlarged dual space the same as $\mathcal{Y}(y)$ defined as the set of supermartingales deflators

$$\mathcal{Y}(y) = \Big\{ Y \geq 0 \Big| Y_0 = y \text{ and } XY = (X_t Y_t)_{0\leq t\leq T} \text{ is a supermartingale} \\ \text{for each } X \in \mathcal{X}(x) \Big\},$$

where $\mathcal{X}(x)$ denotes the set of accumulated gains/losses process under some admissible investment strategies with initial endowment less than or equal to x. See the precise definition in the paper by *Kramkov and Schachermayer* [48], [49], or, in the filtered version by *Žitković* [76]. The main difficulty of our problem lies in the fact that the dual functional is no longer necessarily lower semicontinuous with respect to the supermartingale deflator process Y_t. As a matter of fact, if we formally derive the naive dual problem by using the Legendre transform and the first order condition, we arrive at

$$\inf_{y>0, Y\in\mathcal{Y}(1)} \mathbb{E}\Big[\int_0^T V\Big(yY_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)dv} yY_s ds\Big|\mathcal{F}_t\Big]\Big)dt\Big] \\ - z\mathbb{E}\Big[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv} yY_t dt\Big].$$

The first mathematical difficulty we need to handle is the extra integral $\mathbb{E}\Big[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv} yY_t dt\Big]$. At a first glance, it reminds us to invoke the general treatment of random endowment developed by *Cvitanic, Schachermayer and Wang* [18], *Karatzas and Žitković* [43] and *Žitković* [77]. Their work requires the extension of the set $\mathcal{M}$ of equivalent local martingale measures to the set $\mathcal{D}$ of bounded finitely additive measures. Nevertheless, their approach is inadequate to deal with the first term of the dual problem, when the conditional integral part $\mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)dv} yY_s ds\Big|\mathcal{F}_t\Big]$ in the conjugate function V is taken into account. The analysis becomes more complicated since the conditional expectation is not well defined under finitely additive measures and the primal optimizer will possibly depends on the singular part of some finitely additive measures. This is bad news for our analysis since, on the one hand, the singular part is not unique, and on the other hand, it is too abstract to carry any explicit financial information.

In order to avoid the complexity of the path-dependence and the difficulties stated above, we propose the novel transformation from the consumption rate process c_t to its auxiliary process $\tilde{c} = c_t - Z_t$, so that the primal utility maximization problem becomes time separable with respect to the process $\tilde{c}_t$. This substitution idea from c_t to $\tilde{c}_t$ appeared firstly in the Market Isomorphism result for complete markets by *Schroder and Skiadas* [71]. And meanwhile, for each equivalent local martingale measure density process $Y \in \mathcal{M}$, we define the auxiliary dual process Γ_t exactly by

$$\Gamma_t \triangleq Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} Y_s ds \Big| \mathcal{F}_t\Big], \quad \forall t \in [0, T].$$

We naturally intend to rewrite the dual problem given above in terms of auxiliary process Γ_t instead of Y_t so that the path dependence of Y_t can be also hidden in the definition of process Γ_t.

However, the integral of the extra exogenous random term with respect to dual process Y_t, i.e., $\mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} Y_t dt\Big]$ remains in the formulation of the dual problem. By introducing the stochastic process $\tilde{w}_t = \exp(\int_0^t (-\alpha_v) dv)$, which itself is fully determined by the discounting processes δ_t and α_t, a direct application of the Fubini-Tonelli' theorem brings us the hope that one can shift the integral $\mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} Y_t dt\Big]$ involving process $Y_t \in \mathcal{M}$ to the integral $\mathbb{E}\Big[\int_0^T \tilde{w}_t \Gamma_t dt\Big]$ with respect to its auxiliary process Γ_t. With the aid of this equality, we can naturally treat the extra exogenous random term $\tilde{w}_t$ as some shadow random endowment density process in the abstract product space, and define the dual functional on the properly modified space of Γ_t instead of Y_t. As we pointed out, the approach of finitely additive measures can not supply a pleasant treatment with respect to the conditional expectation term since some unknown singular parts are missing. However, as long as the initial standard of living value z is regarded as the variable of the value function, we can add one more dimension to the conjugate duality results and hide the extra integral term $\mathbb{E}\Big[\int_0^T \tilde{w}_t \Gamma_t dt\Big]$ by controlling its values. In essence, by enlarging the effective domain of values for x and z, we arrange to embed our original utility maximization problem with consumption habit formation into the framework of *Hugonnier and Kramkov* [35] for an abstract time separable utility maximization on the product space.

On the other hand, we are facing issues in trying to apply the classical convex duality results to the auxiliary processes $\tilde{c}_t$ and Γ_t when the dual space may not be a subset of $\mathbb{L}^1(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$, because the auxiliary dual process Γ_t is defined via the unbounded stochastic discounting factors α_t and δ_t. We highlight this relaxation on the dual space needs more assumptions to guarantee that our original optimization problem is well defined and to revise some classical proofs in convex duality which heavily rely on the fact that the dual process is integrable under the original probability measure. For example, the integrability of dual process plays an important role in the classical proofs of existence of dual and primal optimizers and the conjugate duality relation between value functions. See section 3 in *Kramkov and Schachermayer* [48] and Appendix A in *Karatzas and Žitković* [43] for details.

Indeed, we impose the market independent sufficient conditions on habit formation discounting factors α_t and δ_t, see Assumption (2.3.3) and (2.3.4), to guarantee the well-posedness of the Primal optimization problem. We also ask for the Reasonable Asymptotic Elasticity conditions on utility functions U both at $x \to 0$ and $x \to \infty$, i.e., $AE_0[U] < \infty$ and $AE_\infty[U] < 1$ (see Assumption (2.2.9) and (2.2.10)), for the validity of several key assertions of our main results to hold true. To the best of our knowledge, our work is the first one which aims to solve the utility maximization problem with consumption habit formation in continuous time framework in general incomplete semimartingale financial markets.

We should also stress that our work is the first step to study the utility maximization problem with general nonlinear habit formation $\mathbb{E}[\int_0^T U(t, c_t, Z_t)dt]$ in incomplete semimartingale markets, in the sense that the investor's preference depends nonlinearly on both the current consumption rate process c_t and his past consumption path accumulative index Z_t. This generalized nonlinear habit formation problem includes the *non-addictive linear* habits considered earlier by *Detemple and Karatzas* [23]. We intend to provide similar convex duality conclusions as well as some specific characterizations of the optimal consumption structures in future research. Another main motivation behind this work is the role it plays as a necessary step for the existence and uniqueness for equilibrium in continuous-time incomplete markets, together with internal/external habit formation or other time non-separable preferences, see *Detemple and Zapatero* [24] and *Bank and Riedel* [2], [3] for examples in complete markets.

The rest of this chapter is organized in the following way: Section 2.2 first describes the financial market. For the purpose to ensure the original utility optimization problem is well defined and to assist future proofs of the main results, we impose in this section the Reasonable Asymptotic Elasticity Condition of the Utility function both for $x = \infty$ and $x = 0$. In Section 2.3, we introduce some functional set-up on the product space $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$, and define the auxiliary process domain $\bar{\mathcal{A}}(x,z)$ and the auxiliary dual space $\widetilde{\mathcal{M}}$. We embed our original problem into an auxiliary abstract utility maximization problem without habit formation over the enlarged abstract admissible space $\widetilde{\mathcal{A}}(x,z)$, however, with the shadow random endowment. We first assume that the extra exogenous term $\mathcal{E} = \int_0^T w_t dt$ is not replicable under the original market in Section 2.4, and this section is devoted to the definition of the two dimensional dual problem over the properly enlarged dual space $\widetilde{\mathcal{Y}}(y,r)$ for the auxiliary primal optimization problem and our main results are stated in the end. Section 2.5 contains the proofs of our main results. Section 2.6 complement our main results by concerning the special case of replicable extra exogenous term $\mathcal{E}$, and some important and interesting features are exhibited and discussed.

2.2 Market Model

2.2.1 The Financial Market Model

We consider a financial market with $d \in \mathbb{N}$ risky assets modeled by a d-dimensional semimartingale

$$S = (S_t^{(1)}, \ldots, S_t^{(d)})_{t \in [0,T]} \tag{2.2.1}$$

on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$, where the filtration $\mathbb{F}$ satisfies the usual conditions and the maturity time is given by T. To simplify our notation, we take $\mathcal{F} = \mathcal{F}_T$.

We make the standard assumption that there exists one riskless bond $S_t^{(0)} \equiv 1, \forall t \in [0,T]$, which amounts to consider $S_t^{(0)}$ as the numéraire asset.

The portfolio process $H = (H_t^{(1)}, \ldots, H_t^{(d)})_{t \in [0,T]}$ is a predictable S-integrable process representing the number of shares of each risky asset held by the investor at time $t \in [0,T]$. The accumulated gains/losses process of the investor under his trading strategy H by time t is given by:

$$X_t^H = (H \cdot S)_t = \sum_{k=1}^{d} \int_0^t H_u^{(k)} dS_u^{(k)}, \quad t \in [0,T]. \tag{2.2.2}$$

2.2.2 Admissible Portfolios and Consumption Habits

The portfolio process $(H_t)_{t \in [0,T]}$ is called **admissible** if the gains/losses process X_t^H is bounded below, which is to say, there exists a constant bound $a \in \mathbb{R}$ such that $X_t^H \ge a$, $a.s.$ for all $t \in [0,T]$.

Now, given the initial wealth $x > 0$, the agent will also choose an intermediate consumption plan during the whole investment horizon, and we denote the consumption rate process by c_t. The resulting self-financing **wealth process** $(W_t^{x,H,c})_{t\in[0,T]}$ is given by

$$W_t^{x,H,c} \triangleq x + (H \cdot S)_t - \int_0^t c_s ds, \quad t \in [0,T]. \tag{2.2.3}$$

Besides of the wealth process, as we defined in Chapter 1, the associated consumption habit formation process $Z_\cdot \triangleq Z(\cdot; c)$ is given in the following way to align with the previous notations and the literature:

$$\begin{aligned} dZ_t &= (\delta_t c_t - \alpha_t Z_t)dt, \\ Z_0 &= z, \end{aligned}$$

where the stochastic discounting factors α_t and δ_t are assumed to be nonnegative optional processes and the given real number $z \geq 0$ is called *"initial habit"* or *"initial standard of living"*.

Remark 2.2.1. *In this chapter, we shall be mostly interested in the general case when the discounting factors α_t and δ_t are stochastic processes which are allowed to be unbounded. The stochastic nature of the discounting factors corresponds to various market features. For instance, the investor may randomly change his weights on the consumption habits impact due to his risk preference change, time-varying impatience or other time inconsistent incentives from the financial market.*

Equivalently, we can write the habit formation process Z_t as

$$Z_t = ze^{-\int_0^t \alpha_v dv} + \int_0^t \delta_s e^{-\int_s^t \alpha_v dv} c_s ds, \tag{2.2.4}$$

which is also called *"the standard of living"* process and represents the *"Habit Formation"* of the investor, an index as exponentially weighted average of agent's past consumption integral. Here, these stochastic discounting factors α_t and δ_t measure, respectively, the persistence of the initial habits level and the intensity of consumption history.

Throughout this chapter, we make the assumption that the consumption habit is addictive, i.e., $c_t \geq Z_t, \ \ \forall t \in [0,T]$, which is to say, the investor's current consumption rate shall never fall below his *"the standard of living"* process.

A consumption process $(c_t)_{t\in[0,T]}$ is defined to be (x,z)-**financeable** if there exists an admissible portfolio process $(H_t)_{t\in[0,T]}$ such that $W_t^{x,H,c} \geq 0, \ \forall t \in [0,T]$, a.s. and the addictive habit formation constraint $c_t \geq Z_t, \ \forall t \in [0,T]$ a.s. holds. The class of all (x,z)-financeable consumption rate processes will be denoted by $\mathcal{A}(x,z)$, for $x > 0$, $z \geq 0$.

2.2.3 Absence of Arbitrage

A probability measure $\mathbb{Q}$ is called an **equivalent local martingale measure** if it is equivalent to $\mathbb{P}$ and if X_t^H is a local martingale under $\mathbb{Q}$. We denote by $\mathcal{M}$ the family of equivalent local martingale measures and in order to rule out the arbitrage opportunities in the market, we assume that

$$\mathcal{M} \neq \emptyset. \tag{2.2.5}$$

We refer the readers to *Delbaen and Schachermayer* [20] and [21] for a comprehensive discussion and treatment on the topic of no arbitrage.

Define the RCLL process $Y^{\mathbb{Q}}$ by

$$Y_t^{\mathbb{Q}} = \mathbb{E}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|\mathcal{F}_t\Big]$$

for the $\mathbb{Q} \in \mathcal{M}$, then $Y^{\mathbb{Q}}$ is called an equivalent local martingale measure density and we shall always identify the equivalent local martingale measure $\mathbb{Q}$ with its density process $Y^{\mathbb{Q}}$, and with a slight abuse of notation, we denote $\mathcal{M}$ also as the set of all equivalent local martingale density processes.

The celebrated Optional Decomposition Theorem, see *Kramkov* [47], enables us to characterize the (x, z)-financeble consumption process in terms of linear inequalities with respect to $Y_t \in \mathcal{M}$, called **Budget Constraint**, and this serves as an important ingredient in the treatment of our utility maximization problem via convex duality approach.

Proposition 2.2.1. *The process $(c_t)_{t\in[0,T]}$ is (x, z)- financeable if and only if $c_t \geq Z_t,\ \forall t \in [0, T]$ and*

$$\mathbb{E}\Big[\int_0^T c_t Y_t dt\Big] \ \leq \ x, \quad \forall Y_t \in \mathcal{M}. \tag{2.2.6}$$

Remark 2.2.2. *Due to the feature of the path dependent constraint, it is easy to see the set of (x, z)-financeble consumption processes is in general not solid. This means it is generally not allowed if we only choose to consume as less as we want than some admissible strategies since the habit formation constraint may not be retained. Some technical questions in convex duality arise as we expect the solid hull of the admissible set is now generically complicated to describe.*

2.2.4 The Utility Function

The individual investor's preference is represented by a utility function $U : [0,T] \times (0,\infty) \to \mathbb{R}$, such that, for every $x > 0$, $U(\cdot, x)$ is continuous on $[0,T]$, and for every $t \in [0,T]$, the function $U(t,\cdot)$ is strictly concave, strictly increasing, continuously differentiable and satisfies the Inada conditions:

$$U'(t,0) \triangleq \lim_{x\to 0} U'(t,x) = \infty, \qquad U'(t,\infty) \triangleq \lim_{x\to\infty} U'(t,x) = 0. \tag{2.2.7}$$

where $U'(t,x) \triangleq \frac{\partial}{\partial x} U(t,x)$.

According to these assumptions, the inverse $I(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ of the function $U'(t,\cdot)$ exists for every $t \in [0,T]$, and is continuous and strictly decreasing with:

$$I(t,0) \triangleq \lim_{x\to 0} I(t,x) = \infty, \qquad I(t,\infty) \triangleq \lim_{x\to\infty} I(t,x) = 0. \tag{2.2.8}$$

The convex conjugate of the agents' utility function, also known as the Legendre-Fenchel transform, is defined as follows:

$$V(t,y) \triangleq \sup_{x>0}\{U(t,x) - xy\}, \qquad y > 0.$$

Under the Inada conditions (2.2.7), the conjugate of $V(t,\cdot)$ is a continuously differentiable, strictly decreasing and strictly convex function satisfying $V'(t,0) = -\infty$, $V'(t,\infty) = 0$ and $V(t,0) = U(t,\infty)$, $V(t,\infty) = U(t,0)$, see, for example, *Karatzas, Lehoczky, Shreve, and Xu* [40] for reference.

Follow the asymptotic growth control of the utility functions coined by *Kramkov and Schachermayer* [48], see also *Karatzas and Žitković* [43], we shall make additional assumptions on the asymptotic behavior of U at both $x = 0$ and $x = \infty$ for future purposes:

Assumption 2.2.1.

The utility functions U satisfies the Reasonable Asymptotic Elasticity condition both at $x = \infty$ and $x = 0$, i.e.,

$$AE_\infty[U] = \limsup_{x\to\infty}\Big(\sup_{t\in[0,T]} \frac{x\,U'(t,x)}{U(t,x)}\Big) < 1, \tag{2.2.9}$$

and

$$AE_0[U] = \limsup_{x\to 0}\Big(\sup_{t\in[0,T]} \frac{x\,U'(t,x)}{|U(t,x)|}\Big) < \infty. \tag{2.2.10}$$

Moreover, in order to get some inequalities uniformly in time t, we shall assume

$$\lim_{x\to\infty}\Big(\inf_{t\in[0,T]} U(t,x)\Big) > 0, \tag{2.2.11}$$

and

$$\lim_{x\to 0}\Big(\sup_{t\in[0,T]} U(t,x)\Big) < 0. \tag{2.2.12}$$

Remark 2.2.3. *Many well known Utility functions satisfy Reasonable Asymptotic Elasticity Assumptions* (2.2.9) *and* (2.2.10)*, for example, the discounted log utility function $U(t,x) = e^{-\beta t}\log(x)$ or discounted power utility function $U(t,x) = e^{-\beta t}\frac{x^p}{p}$ ($p < 1$ and $p \neq 0$), for a constant $\beta > 0$. However, it is also easy to check that the utility function $U(t,x) = -e^{\frac{1}{x}}$ does not satisfy the Assumption* (2.2.10) *and the utility function $U(t,x) = \frac{x}{\log x}$ does not satisfy the Assumption* (2.2.9).

Remark 2.2.4. *If the Utility function satisfies the lower bound assumption* $\inf_{t\in[0,T]} U(t,0) > -\infty$*, then our Assumption* (2.2.10) *is automatically verified. And if the Utility function satisfies the upper bound assumption* $\sup_{t\in[0,T]} U(t,\infty) < \infty$*, the Assumption* (2.2.9) *holds true.*

Remark 2.2.5. *The utility function* $U(t,x)$ *satisfies Reasonable Asymptotic Elasticity Assumptions* (2.2.9) *and* (2.2.10) *if and only if its affine transform* $a + bU(t,x)$ *satisfies Reasonable Asymptotic Elasticity Assumptions* (2.2.9) *and* (2.2.10) *for arbitrary constants* $a, b > 0$*. Hence, the adjoint Assumption* (2.2.11) *and Assumption* (2.2.12) *are not restrictive.*

The next technical result gives the equivalent characterization of the Reasonable Asymptotic Elasticity condition $AE_\infty[U]$, which follows the identical proof of Lemma 6.3 of *Kramkov and Schachermayer* [48], see also Proposition 3.7 of *Karatzas and Žitković* [43].

Lemma 2.2.2. *Let* $U(t,x)$ *be a utility function satisfying* (2.2.9) *and* (2.2.11)*. In each of the subsequent assertions, the infimum of* $\gamma > 0$ *for which these assertions hold true equals the Reasonable Asymptotic Elasticity* $AE_\infty[U]$*.*

(i) There is $x_0 > 0$ *for all* $t \in [0,T]$ *s.t.*

$$U(t,\lambda x) < \lambda^\gamma U(t,x) \quad \textit{for } \lambda > 1, x \geq x_0.$$

(ii) There is $x_0 > 0$ *for all* $t \in [0,T]$ *s.t.*

$$U'(t,x) < \gamma \frac{U(t,x)}{x} \quad \textit{for } x \geq x_0.$$

(iii) There is $y_0 > 0$ for all $t \in [0,T]$ s.t.

$$V(t,\mu y) < \mu^{-\frac{\gamma}{1-\gamma}} V(t,y) \quad \text{for } 0 < \mu < 1, 0 < y \leq y_0.$$

(iv) There is $y_0 > 0$ for all $t \in [0,T]$ s.t.

$$-V'(t,y) < \Big(\frac{\gamma}{1-\gamma}\Big)\frac{V(t,y)}{y} \quad \text{for } 0 < y \leq y_0.$$

Corollary 2.2.3. *Under Assumptions (2.2.10) and (2.2.12), we have $AE_0[U] < \infty$ if and only if $AE_\infty[V] < 1$, where we define*

$$AE_\infty[V] = \limsup_{y\to\infty}\Big(\sup_{t\in[0,T]}\frac{y\,V'(t,y)}{V(t,y)}\Big) < 1, \tag{2.2.13}$$

and hence similarly, we have each of the following assertions, the infimum of $\gamma > 0$ for which these assertions hold true equals the Reasonable Asymptotic Elasticity $AE_\infty[V]$.

(i) There is $y_0 > 0$ for all $t \in [0,T]$ s.t.

$$V(t,\lambda y) > \lambda^\gamma V(t,y) \quad \text{for } \lambda > 1, y \geq y_0.$$

(ii) There is $y_0 > 0$ for all $t \in [0,T]$ s.t.

$$V'(t,y) > \gamma\frac{V(t,y)}{y} \quad \text{for } y \geq y_0.$$

(iii) There is $x_0 > 0$ for all $t \in [0,T]$ s.t.

$$U(t,\mu x) > \mu^{-\frac{\gamma}{1-\gamma}} U(t,x) \quad \text{for } 0 < \mu < 1, 0 < x \leq x_0.$$

(iv) There is $x_0 > 0$ for all $t \in [0,T]$ s.t.

$$-U'(t,x) > \Big(\frac{\gamma}{1-\gamma}\Big)\frac{U(t,x)}{x} \quad \text{for } 0 < x \leq x_0.$$

2.3 A New Characterization of Financeable Consumption Processes

2.3.1 Some Functional Set Up

In the spirit of *Bouchard and Pham* [10] who treats the wealth dependent problem (see also *Žitković* [77] on consumption and endowment with stochastic clock), let $\mathcal{O}$ denotes the σ-algebra of optional sets relative to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and we define the product measure $d\bar{\mathbb{P}} = dt \times d\mathbb{P}$ be the finite measure on the product space $(\Omega \times [0,T], \mathcal{O})$:

$$\bar{\mathbb{P}}[A] = \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \mathbf{1}_A(t,\omega)dt\Big], \quad \text{for } A \in \mathcal{O}. \tag{2.3.1}$$

We denote by $\mathbb{L}^0(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$ ($\mathbb{L}^0$ for short) the set of all random variables on the product space $\Omega \times [0,T]$ under the product measure $\bar{\mathbb{P}}$ with respect to the optional σ-algebra $\mathcal{O}$ endowed with the topology of convergence in measure $\bar{\mathbb{P}}$. And from now on, we shall always identify the optional stochastic process $(Y_t)_{t\in[0,T]}$ with the random variable $Y \in \mathbb{L}^0(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$. We also define the positive orthant $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$ ($\mathbb{L}^0_+$ for short) the set of elements $Y = Y(t,\omega)$ of $\mathbb{L}^0$ such that:

$$Y \geq 0, \quad \bar{\mathbb{P}}\ a.s..$$

For any $Y^1, Y^2 \in \mathbb{L}^0_+$, we shall say that

$$Y^1 \equiv Y^2 \ \text{ if } \ Y^1 = Y^2, \ \ \bar{\mathbb{P}}\ a.s..$$

Endow $\mathbb{L}^0_+$ with the bilinear form valued in $[0,\infty]$ as:

$$\big\langle X, Y \big\rangle = \mathbb{E}\Big[\int_0^T X_t Y_t dt\Big], \ \text{ for all } X, Y \in \mathbb{L}^0_+.$$

We also define a partial ordering on $\mathbb{L}^0_+$ for convenience:

$$Y^1 \preceq (\prec) Y^2 \Longleftrightarrow Y^1 \leq (<) Y^2, \quad \bar{\mathbb{P}}\ a.s..$$

2.3.2 Path-dependence Reduction by Auxiliary Processes

At this point, we are able to define the set of all (x,z)-financeable consumption rate processes as a set of random variables on the product space $(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$ and the **Budget Constraint** Proposition 2.2.1 states that:

$$\begin{aligned}\mathcal{A}(x,z) \triangleq \Big\{ & c \in \mathbb{L}^0_+ : c_t \geq Z_t \text{ and } W_t = x + \\ & (H \cdot S)_t - \int_0^t c_s ds \geq 0, \forall t \in [0,T] \text{ and } H \text{ is admissible} \Big\} \\ = \Big\{ & c \in \mathbb{L}^0_+ : c_t \geq Z_t,\ \forall t \in [0,T] \text{ and } \big\langle c, Y \big\rangle \leq x,\ \forall Y \in \mathcal{M} \Big\}.\end{aligned}$$

where process Z_t is defined by (2.2.4). However, the family $\mathcal{A}(x,z)$ may be empty for some values $x > 0, z \geq 0$. We shall restrict ourselves to the *effective domain* $\bar{\mathcal{H}}$ which is defined as the union of the *interior* of set such that $\mathcal{A}(x,z)$ is not empty and the one side boundary $\{x > 0, z = 0\}$:

$$\bar{\mathcal{H}} \triangleq int\Big\{(x,z) \in (0,\infty) \times [0,\infty) : \mathcal{A}(x,z) \neq \varnothing\Big\} \cup (0,\infty) \times \{0\}. \qquad (2.3.2)$$

We want the effective domain $\bar{\mathcal{H}}$ to include the special case of zero initial habit by $z = 0$.

Before we state the next result, we shall first impose some additional conditions on the stochastic discounting factors α_t and δ_t, which are essential for the well-posedness of our primal utility optimization problem :

Assumption 2.3.1.

We assume the nonnegative optional processes α_t *and* δ_t *satisfy:*

$$\sup_{Y \in \mathcal{M}} \mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} Y_t dt\Big] < \infty. \tag{2.3.3}$$

and there exists a constant $\bar{x} > 0$ *such that*

$$\mathbb{E}\Big[\int_0^T U(t, \bar{x} e^{-\int_0^t \alpha_v dv}) dt\Big] > -\infty. \tag{2.3.4}$$

Remark 2.3.1. *If stochastic discounting processes* α_t *and* δ_t *are assumed to be bounded, Assumptions* (2.3.3) *and* (2.3.4) *will be satisfied, and are redundant.*

Remark 2.3.2. *Assumption* (2.3.3) *is the well known super-hedging property of the random variable* $\int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} dt$ *in our original financial market.*

On the other hand, we make Assumption (2.3.4) *to guarantee the existence of some* $(x, z) \in \bar{\mathcal{H}}$ *such that the value function* $u(x, z) > -\infty$*, which is always taken as granted in the utility maximization problem with pure investment or consumption without habit formation. The acceptance of this convention in the classical problem lies in the fact that there exists some strict positive constants in the corresponding admissible space of wealth or consumption processes. However, this convention will be violated in the context of consumption habit formation. It is interesting to note, however, in the future we will see the process* $\tilde{w}_t \triangleq e^{-\int_0^t \alpha_v dv}$ *somehow plays the same role as the constant* 1 *to be a universal strictly positive element in the corresponding admissible space by rescaling. And we remark here that one can also take* $\tilde{w}_t \triangleq e^{-\int_0^t \alpha_v dv}$ *as the abstract numeráre.*

Lemma 2.3.1. *Under Assumption* (2.3.3)*, the effective domain* $\bar{\mathcal{H}}$ *can be rewritten explicitly as:*

$$\bar{\mathcal{H}} = \Big\{(x,z) \in (0,\infty) \times [0,\infty) : x > z \sup_{Y \in \mathcal{M}} \mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_t dt\Big]\Big\}. \quad (2.3.5)$$

Proof. It is enough to show for all $(x,z) \in (0,\infty) \times [0,\infty)$,

$$x \geq z \sup_{Y \in \mathcal{M}} \mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} Y_t dt\Big] \quad (2.3.6)$$

if and only if $\mathcal{A}(x,z) \neq \emptyset$.

On one hand, if $(x,z) \in (0,\infty) \times [0,\infty)$ and $\mathcal{A}(x,z) \neq \emptyset$, by definition, there exists $c \in \mathbb{L}^0_+$ such that $c_t \geq Z_t$ for all $t \in [0,T]$ and $\langle c, Y \rangle \leq x, \forall Y \in \mathcal{M}$. We now claim that we should always have $c_t \geq \bar{c}_t$ for all $t \in [0,T]$ where $\bar{c}_t \equiv Z(\bar{c})_t$ is the subsistent consumption plan which equals its standard of living process. To this end, we first recall by the definition of Z_t that $dZ_t = (\delta_t c_t - \alpha_t Z_t)dt$ with $Z_0 = z \geq 0$, and the constraint that $c_t \geq Z_t$ implies

$$dZ_t \geq (\delta_t Z_t - \alpha_t Z_t)dt, \quad Z_0 = z, \quad (2.3.7)$$

also, we should have $\bar{c}_t$ satisfies

$$d\bar{c}_t = (\delta_t \bar{c}_t - \alpha_t \bar{c}_t)dt, \quad \bar{c}_0 = z. \quad (2.3.8)$$

and we can solve $\bar{c}_t = z e^{\int_0^t (\delta_v - \alpha_v)dv}$ for $t \in [0,T]$.

By the simple subtraction of (2.3.7) and (2.3.8), one can get

$$d(Z_t - \bar{c}_t) \geq (\delta_t - \alpha_t)(Z_t - \bar{c}_t)dt, \quad Z_0 - \bar{c}_0 = 0,$$

from which we can derive that

$$e^{\int_0^t (\delta_s - \alpha_s) ds}(Z_t - \bar{c}_t) \geq 0, \quad \forall t \in [0, T]. \tag{2.3.9}$$

Hence, we will conclude that $c_t \geq z e^{\int_0^t (\delta_v - \alpha_v) dv}$ for all $t \in [0, T]$, which gives $x \geq z \sup_{Y \in \mathcal{M}} \mathbb{E}\Big[\int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} Y_t dt \Big]$ by the consumption Budget Constraint condition (2.2.6).

One the other hand, if $(x, z) \in (0, \infty) \times [0, \infty)$ and (2.3.6) holds, we can obviously always construct $\bar{c}_t = z e^{\int_0^t (\delta_v - \alpha_v) dv}$ such that $\bar{c}_t \equiv Z(\bar{c})_t$ for all $t \in [0, T]$ and $\langle c, Y \rangle \leq x, \forall Y \in \mathcal{M}$, and hence $\mathcal{A}(x, z) \neq \emptyset$. The proof is complete. $\square$

By choosing $(x, z) \in \bar{\mathcal{H}}$, we can now define the preliminary version of our **Primal Utility Maximization Problem** as:

$$u(x, z) \triangleq \sup_{c \in \mathcal{A}(x,z)} \mathbb{E}\Big[\int_0^T U(t, c_t - Z_t) dt \Big], \quad (x, z) \in \bar{\mathcal{H}}. \tag{2.3.10}$$

Now, for fixed $(x, z) \in \bar{\mathcal{H}}$, and each (x, z)-financeable consumption rate process, we want to generalize the Market Isomorphism idea by *Schroder and Skiadas* [71] in order to reduce the path dependency. We are ready to introduce the auxiliary process $\tilde{c}_t = c_t - Z_t$, and define the auxiliary set of $\mathcal{A}(x, z)$ as:

$$\bar{\mathcal{A}}(x, z) \triangleq \Big\{ \tilde{c} \in \mathbb{L}^0_+ : \ \tilde{c}_t = c_t - Z_t, \ \forall t \in [0, T], \ \ c \in \mathcal{A}(x, z) \Big\}. \tag{2.3.11}$$

Lemma 2.3.2. *For each fixed $(x, z) \in \bar{\mathcal{H}}$, there is a one to one correspondence between sets $\mathcal{A}(x, z)$ and $\bar{\mathcal{A}}(x, z)$, and hence we have $\bar{\mathcal{A}}(x, z) \neq \emptyset$ for $(x, z) \in \bar{\mathcal{H}}$.*

Proof. Fix each pair $(x, z) \in \bar{\mathcal{H}}$ so that $\mathcal{A}(x, z) \neq \emptyset$, it is clear by the definition that for each $c \in \mathcal{A}(x, z)$, there exists a unique $\tilde{c}_t = c_t - Z_t$ such that $\tilde{c} \in \bar{\mathcal{A}}(x, z)$.

Now for each fixed $(x, z) \in \bar{\mathcal{H}}$ and $\tilde{c} \in \bar{\mathcal{A}}$, denote the process

$$c_t \triangleq \tilde{c}_t + \widetilde{Z}_t,$$

where the process $\widetilde{Z}_t$ is uniquely determined by the process $\tilde{c}$ as

$$\widetilde{Z}_t = ze^{\int_0^t (\delta_v - \alpha_v) dv} + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds.$$

It is easy to check by definition that $c_t - Z_t = \tilde{c}_t \geq 0$, where we know

$$Z_t = ze^{-\int_0^t \alpha_v dv} + \int_0^t \delta_s e^{-\int_s^t \alpha_v dv} c_s ds.$$

Now by the definition of set $\bar{\mathcal{A}}(x, z)$ and the uniqueness of process c_t such that $c_t - Z_t = \tilde{c}_t$, we can therefore conclude there exists a unique $c_t \in \mathcal{A}(x, z)$ for each $\tilde{c} \in \bar{\mathcal{A}}(x, z)$. $\square$

Let's turn our attention to the set $\mathcal{M}$ of equivalent local martingale measures, and for each $Y \in \mathcal{M}$, according to Assumption (2.3.3) we can define the auxiliary optional process with respect to Y_t as:

$$\Gamma_t \triangleq Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} Y_s ds \Big| \mathcal{F}_t\Big], \quad \forall t \in [0, T]. \tag{2.3.12}$$

Let's denote the set of all these auxiliary optional processes as:

$$\widetilde{\mathcal{M}} = \Big\{\Gamma \in \mathbb{L}_+^0 : \Gamma_t = Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} Y_s ds \Big| \mathcal{F}_t\Big], \ \ \forall t \in [0, T], \ \ Y \in \mathcal{M}\Big\}. \tag{2.3.13}$$

We remark again here that since stochastic discounting processes δ_t and α_t are unbounded, under Assumption (2.3.3), the auxiliary dual process Γ is well defined, but it is not necessarily in $\mathbb{L}^1$.

The following important equalities serve as critical ingredients in embedding our original utility maximization problem into an auxiliary abstract optimization problem on the product space, for which we are able to apply the convex duality approach:

Proposition 2.3.3. *Under Assumption* (2.3.3)*, for each nonnegative optional process c_t such that $c_t \geq Z_t$ with Z_t defined by* (2.2.4) *for fixed initial standard of living $z \geq 0$ and the nonnegative optional process Y_t, we have the following equalities with respect to their corresponding auxiliary processes $\tilde{c}_t = c_t - Z_t$ and Γ_t which is defined by* (2.3.12)*, that:*

$$\begin{aligned} \big\langle c, Y \big\rangle &= \big\langle \tilde{c}, \Gamma \big\rangle + z\big\langle w, Y \big\rangle \\ &= \big\langle \tilde{c}, \Gamma \big\rangle + z\big\langle \tilde{w}, \Gamma \big\rangle, \end{aligned} \tag{2.3.14}$$

where we define these extra exogenous random processes $w,\ \tilde{w} \in \mathbb{L}^0_+$ as

$$w_t \triangleq e^{\int_0^t (\delta_v - \alpha_v)dv} \quad \text{and} \quad \tilde{w}_t \triangleq e^{\int_0^t (-\alpha_v)dv} \ \text{for all } t \in [0, T]. \tag{2.3.15}$$

Proof. By the definition, Z_t solves the ODE: $dZ_t = (\delta_t c_t - \alpha_t Z_t)dt$ with $Z_0 = z$, for each $t \in [0, T]$. If we set $\tilde{c}_t = c_t - Z_t$, we can rewrite c_t in terms of $\tilde{c}_t$ as:

$$c_t = z e^{\int_0^t (\delta_v - \alpha_v)dv} + \tilde{c}_t + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v)dv} \tilde{c}_s ds,$$

and hence we will have the following chain equivalence by Tonelli's theorem:

$$
\begin{aligned}
\big\langle c, Y\big\rangle =& z\mathbb{E}\Big[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv}Y_t dt\Big] + \mathbb{E}\Big[\int_0^T \Big(\tilde{c}_t + \int_0^t \delta_s e^{\int_s^t(\delta_v-\alpha_v)dv}\tilde{c}_s ds\Big)Y_t dt\Big]\\
=& z\big\langle w, Y\big\rangle + \mathbb{E}\Big[\int_0^T \tilde{c}_t Y_t dt + \int_0^T \delta_s \tilde{c}_s\Big(\int_s^T e^{\int_s^t(\delta_v-\alpha_v)dv}Y_t dt\Big)ds\Big]\\
=& z\big\langle w, Y\big\rangle + \mathbb{E}\Big[\int_0^T \tilde{c}_t Y_t dt + \int_0^T \delta_t \tilde{c}_t \mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)dv}Y_s ds\Big|\mathcal{F}_t\Big]dt\Big]\\
=& z\big\langle w, Y\big\rangle + \big\langle \tilde{c}, \Gamma\big\rangle,
\end{aligned}
$$

which gives the first equality. Similarly, we just observe that:

$$
\begin{aligned}
\big\langle \tilde{w}, \Gamma\big\rangle =& \mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv}Y_t dt\Big] + \mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv}\delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)dv}Y_s ds\Big|\mathcal{F}_t\Big]dt\Big]\\
=& \mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv}Y_t dt\Big] + \mathbb{E}\Big[\int_0^T e^{\int_0^s(\delta_v-\alpha_v)dv}Y_s\Big(\int_0^s \delta_t e^{-\int_0^t \delta_v dv}dt\Big)ds\Big]\\
=& \mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv}Y_t dt\Big] - \mathbb{E}\Big[\int_0^T e^{\int_0^s(\delta_v-\alpha_v)dv}Y_s\Big(e^{-\int_0^s \delta_s ds} - 1\Big)ds\Big]\\
=& \mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv}Y_t dt\Big] - \mathbb{E}\Big[\int_0^T e^{\int_0^t(-\alpha_v)dv}Y_t dt\Big] + \mathbb{E}\Big[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)dv}Y_t dt\Big]\\
=& \big\langle w, Y\big\rangle,
\end{aligned}
$$

which gives the second equality. □

Remark 2.3.3. *These extra random processes w_t and $\tilde{w}_t$ in* (2.3.15) *defined by stochastic discounting factors α_t and δ_t will play the role of shadow random endowment rate processes in the future formulation of the dual optimization problem. In an attempt to analyze this special structure, we will naturally adopt some classical convex duality analysis with respect to market random endowment source, and try to prove some similar results.*

Based on previous Propositions 2.2.1 and 2.3.3, under Assumptions

(2.3.3) and (2.3.4), clearly we will have the alternative budget constraint characterization of the consumption rate process c_t as:

Proposition 2.3.4. *For any given pair $(x, z) \in \bar{\mathcal{H}}$, we call the consumption rate process c is (x, z)-financeable if and only if $c_t \geq Z_t,\ \forall t \in [0, T]$ and*

$$\left\langle c - Z, \Gamma \right\rangle \leq x - z\left\langle \tilde{w}, \Gamma \right\rangle,\ \textit{for all}\ \Gamma \in \widetilde{\mathcal{M}}.$$

Proposition 2.3.4 provides us the alternative definition of set $\bar{\mathcal{A}}(x, z)$ for $(x, z) \in \bar{\mathcal{H}}$ as:

$$\bar{\mathcal{A}}(x, z) = \Big\{ \tilde{c} \in \mathbb{L}^0_+ :\ \left\langle \tilde{c}, \Gamma \right\rangle \leq x - z\left\langle \tilde{w}, \Gamma \right\rangle,\ \forall \Gamma \in \widetilde{\mathcal{M}} \Big\}. \tag{2.3.16}$$

We see that the path-dependent addictive habits constraint on c_t such that $c_t \geq Z_t$ eventually turns to be a natural constraint that $\tilde{c} \in \mathbb{L}^0_+$, and (2.3.16) states that the auxiliary set $\bar{\mathcal{A}}(x, z)$ is solid, convex and closed in measure $\bar{\mathbb{P}}$ although $\mathcal{A}(x, z)$ does not hold all these properties. Hence this path-dependence reduction from c_t to $\tilde{c}_t$ is crucial to enable us to work with convex duality approach.

2.3.3 Embedding into an Abstract Utility Maximization Problem with the Shadow Random Endowment

In order to apply the classical convex duality analysis for the random endowment and build conjugate duality relations between value functions in the next section, due to some technical reasons, we need to first enlarge the domain of the set $\bar{\mathcal{H}}$ to $\mathcal{H}$ and enlarge the corresponding auxiliary set $\bar{\mathcal{A}}(x, z)$

to $\widetilde{\mathcal{A}}(x,z)$ defined as:

$$\widetilde{\mathcal{A}}(x,z) \triangleq \Big\{\tilde{c} \in \mathbb{L}^0_+ : \big\langle \tilde{c}, \Gamma \big\rangle \leq x - z\big\langle \tilde{w}, \Gamma \big\rangle, \ \forall \Gamma \in \widetilde{\mathcal{M}} \Big\}, \tag{2.3.17}$$

where now $(x,z) \in \mathbb{R}^2$, and is restricted in the enlarged domain $\mathcal{H}$:

$$\mathcal{H} \triangleq int\Big\{(x,z) \in \mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z) \neq \emptyset\Big\}.$$

Under Assumption (2.3.3) and Proposition 2.3.3, we have the following equivalent characterization of $\widetilde{\mathcal{A}}(x,z)$:

Lemma 2.3.5.

$$\begin{aligned} \mathcal{H} =& \Big\{(x,z) \in \mathbb{R}^2 : x > z\big\langle \tilde{w}, \Gamma \big\rangle, \ \textit{for all } \Gamma \in \widetilde{\mathcal{M}}\Big\} \\ =& \Big\{(x,z) \in \mathbb{R}^2 : x > \bar{p}z, z \geq 0\Big\} \cup \Big\{(x,z) \in \mathbb{R}^2 : x > \underline{p}z, z < 0\Big\}. \end{aligned} \tag{2.3.18}$$

where

$$\bar{p} \triangleq \sup_{Y \in \mathcal{M}} \big\langle w, Y \big\rangle = \sup_{\Gamma \in \widetilde{\mathcal{M}}} \big\langle \tilde{w}, \Gamma \big\rangle, \tag{2.3.19}$$

and

$$\underline{p} \triangleq \inf_{Y \in \mathcal{M}} \big\langle w, Y \big\rangle = \inf_{\Gamma \in \widetilde{\mathcal{M}}} \big\langle \tilde{w}, \Gamma \big\rangle. \tag{2.3.20}$$

where $\bar{p}, \underline{p} < \infty$ *and* $\mathcal{H}$ *is a well defined convex cone in* $\mathbb{R}^2$*. Moreover*

$$\begin{aligned} cl\mathcal{H} =& \Big\{(x,z) \in \mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z) \neq \emptyset\Big\} \\ =& \Big\{(x,z) \in \mathbb{R}^2 : x \geq z\big\langle \tilde{w}, \Gamma \big\rangle, \ \textit{for all } \Gamma \in \widetilde{\mathcal{M}}\Big\} \end{aligned} \tag{2.3.21}$$

where $cl\mathcal{H}$ *denotes the closure of the set* $\mathcal{H}$ *in* $\mathbb{R}^2$*.*

Proof. Again, it is just enough to show $\Big\{(x,z) \in \mathbb{R}^2 : x \geq z\big\langle \tilde{w}, \Gamma \big\rangle, \text{ for all } \Gamma_t \in \widetilde{\mathcal{M}}\Big\}$ is equivalent to $\Big\{(x,z) \in \mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z) \neq \emptyset\Big\}$.

On one hand, if $(x,z) \in \Big\{(x,z) \in \mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z) \neq \emptyset\Big\}$, there exists $\tilde{c} \in \mathbb{L}^0_+$ such that $\langle \tilde{c}, \Gamma\rangle \leq x - z\langle \tilde{w}, \Gamma\rangle$ for all $\Gamma \in \widetilde{\mathcal{M}}$, clearly, we get $x \geq z\big\langle \tilde{w}, \Gamma\big\rangle$, for all $\Gamma \in \widetilde{\mathcal{M}}$.

On the other hand, if $(x,z) \in \Big\{(x,z) \in \mathbb{R}^2 : x \geq z\big\langle \tilde{w}, \Gamma\big\rangle,$ for all $\Gamma_t \in \widetilde{\mathcal{M}}\Big\}$, it is trivial to construct $\tilde{c}_t \equiv 0 \in \widetilde{\mathcal{A}}(x,z)$ for all $t \in [0,T]$, therefore, we have $(x,z) \in \Big\{(x,z) \in \mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z) \neq \emptyset\Big\}$, which completes the proof. $\square$

We will now define the **Auxiliary Primal Utility Maximization Problem** based on the abstract auxiliary domain $\widetilde{\mathcal{A}}(x,z)$ as:

$$\tilde{u}(x,z) \triangleq \sup_{\tilde{c} \in \widetilde{\mathcal{A}}(x,z)} \mathbb{E}\Big[\int_0^T U(t, \tilde{c}_t)dt\Big], \ (x,z) \in \mathcal{H}. \tag{2.3.22}$$

By definitions of $\bar{\mathcal{A}}(x,z)$ for $(x,z) \in \bar{\mathcal{H}}$ and $\widetilde{\mathcal{A}}(x,z)$ for $(x,z) \in \mathcal{H}$, we successfully embedded our original utility maximization problem (2.3.10) with consumption habit formation into the auxiliary abstract utility maximization problem (2.3.22) without habit formation, however, with some shadow random endowments. More precisely, the following equivalence can be guaranteed that for any $(x,z) \in \bar{\mathcal{H}} \subset \mathcal{H}$:

$$\bar{\mathcal{A}}(x,z) = \widetilde{\mathcal{A}}(x,z), \tag{2.3.23}$$

and the two value functions coincide

$$u(x,z) = \tilde{u}(x,z), \tag{2.3.24}$$

in addition, the immediate byproduct consequence states that c^*_t is the optimal solution for $u(x,z)$ if and only if $\tilde{c}^*_t = c^*_t - Z^*_t \geq 0$ for all $t \in [0,T]$ is the optimal solution for $\tilde{u}(x,z)$, when $(x,z) \in \bar{\mathcal{H}}$.

2.4 The Dual Optimization Problem and Main Results

Inspired by the idea in *Hugonnier and Kramkov* [35] for optimal investment with random endowment, we concentrate now on the construction of the dual problem by first introducing the set $\mathcal{R}$, which is the *relative interior* of the polar cone of $-\mathcal{H}$:

$$\mathcal{R} \triangleq ri\Big\{(y,r) \in \mathbb{R}^2 : xy - zr \geq 0 \text{ for all } (x,z) \in \mathcal{H}\Big\}. \tag{2.4.1}$$

Let's make the following assumption on stochastic discounting processes α_t and δ_t:

Assumption 2.4.1.

The random variable defined by

$$\mathcal{E} \triangleq \int_0^T w_t dt = \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} dt \tag{2.4.2}$$

is not replicable under our original financial market.

Remark 2.4.1. *Under our Assumption* (2.4.2)*, the existence of Market Isomorphism by Schroder and Skiadas [71] may no longer hold and our work generally extends their conclusion and provides the existence and uniqueness of the optimal solution in incomplete markets using convex duality.*

Remark 2.4.2. *We remark here that even if* $\mathcal{E} \triangleq \int_0^T w_t dt$ *is replicable in the original incomplete market such that* $\bar{p} = \underline{p}$*, the market isomorphism relation* *by Schroder and Skiadas [71] may still not hold. In this case, however, the original utility maximization problem becomes easier since we do not need to take care of the exogenous term $\tilde{w}_t$ and the primal value function $\tilde{u}$ becomes one dimensional function. The special case is discussed in detail in the final section* 2.6.

Lemma 2.4.1. *By Assumption* (2.4.2)*, we know that $\mathcal{R}$ is an open convex cone in $\mathbb{R}^2$, and can be rewritten as:*

$$\mathcal{R} = \Big\{(y,r) \in \mathbb{R}^2 : y > 0, \ \textit{and} \ \ \underline{p}y < r < \bar{p}y\Big\}, \tag{2.4.3}$$

where $\bar{p}$ and $\underline{p}$ are defined by (2.3.19) *and* (2.3.20)*, and $\bar{p} < \underline{p}$.*

Proof. Since $\underline{p} < \bar{p}$ by Assumption (2.4.2), and by Lemma 2.3.5 the set $cl\mathcal{H} = \{(x,z) \in \mathbb{R}^2 : x \geq \bar{p}z, z \geq 0\} \cup \{(x,z) \in \mathbb{R}^2 : x \geq \underline{p}z, z < 0\}$ does not contain any lines passing through the origin. By the properties of polars of convex sets (See *Rockafellar* [66], Corollary 14.6.1), $\mathcal{R}$ is an open convex cone in the first orthant of $\mathbb{R}^2$. Moreover, by the inequality constraint $xy - zr \geq 0$ for all $(x,z) \in \mathcal{H}$ and the definition of $\mathcal{H}$, it is obvious that (2.4.3) holds. □

For an arbitrary pair $(y,r) \in \mathcal{R}$, we denote by $\widetilde{\mathcal{Y}}(y,r)$ the set of non-negative processes as a proper extension of the auxiliary set $\widetilde{\mathcal{M}}$ in the way that:

$$\widetilde{\mathcal{Y}}(y,r) \triangleq \Big\{\Gamma \in \mathbb{L}^0_+ : \ \big\langle \tilde{c}, \Gamma \big\rangle \leq xy - zr, \ \text{for all } \tilde{c} \in \widetilde{\mathcal{A}}(x,z), \ \text{and } (x,z) \in \mathcal{H}\Big\}. \tag{2.4.4}$$

Based on previous efforts, we are ready to establish the **Auxiliary Dual Utility Maximization Problem** to (2.3.22) defined as:

$$\tilde{v}(y,r) \triangleq \inf_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big], \quad (y,r) \in \mathcal{R}. \tag{2.4.5}$$

The following theorems constitute our main results. And we provide their proofs through a number of auxiliary results in the next section.

Theorem 2.4.2. *Assume conditions* (2.2.5)*,* (2.2.7)*,* (2.3.3)*,* (2.3.4)*,* (2.4.2) *hold true. Assume also that* (2.2.11)*,* (2.2.12) *and* (2.2.10) *(i.e.,* $AE_0[U] < \infty$*) together with*

$$\tilde{u}(x,z) < \infty \qquad for\ some\ (x,z) \in \mathcal{H}. \tag{2.4.6}$$

we will have:

(i) The function $\tilde{u}$ is $(-\infty,\infty)$-valued on $\mathcal{H}$ and $\tilde{v}(y,r)$ is $(-\infty,\infty]$-valued on $\mathcal{R}$. And for each $(y,r) \in \mathcal{R}$ there exists a constant $s = s(y,r) > 0$ such that $\tilde{v}(sy,sr) < \infty$. Moreover, we have the conjugate duality of value functions $\tilde{u}$ and $\tilde{v}$:

$$\tilde{u}(x,z) = \inf_{(y,r)\in\mathcal{R}} \{\tilde{v}(y,r) + xy - zr\}, \qquad (x,z) \in \mathcal{H},$$
$$\tilde{v}(y,r) = \sup_{(x,z)\in\mathcal{H}} \{\tilde{u}(x,z) - xy + zr\}, \qquad (y,r) \in \mathcal{R}.$$

(ii) The solution $\Gamma^*(y,r)$ to the optimization problem (2.4.5) *exists and is unique (in the sense of $\equiv$ in $\mathbb{L}^0_+$) for all $(y,r) \in \mathcal{R}$ such that $\tilde{v}(y,r) < \infty$.*

Theorem 2.4.3. *We now assume in addition to conditions of Theorem* 2.4.2 *that Assumption* (2.2.9) *(i.e.,* $AE_\infty[U] < 1$*) holds. Then in addition to assertions of Theorem* 2.4.2*, we also have:*

(i) The value function $\tilde{v}(y,r)$ *is* $(-\infty,\infty)$*-valued on* $(y,r)\in\mathcal{R}$ *and* $\tilde{v}$ *is continuously differentiable on* $\mathcal{L}$*.*

(ii) The solution $\tilde{c}^*(x,z)$ *to optimization problem* (2.3.22) *exists and is unique (in the sense of* $\equiv$ *in* $\mathbb{L}^0_+$*) for any* $(x,z)\in\mathcal{H}$*. And there exists a representation of the optimal solution such that* $\tilde{c}^*_t(x,z)>0$*,* $\mathbb{P}$*-a.s. for all* $t\in[0,T]$*.*

(iii) The superdifferential of $\tilde{u}$ *maps* $\mathcal{H}$ *into* $\mathcal{R}$*, i.e.,*

$$\partial\tilde{u}(x,z)\subset\mathcal{R},\qquad (x,z)\in\mathcal{H}. \tag{2.4.7}$$

Moreover, if $(y,r)\in\partial\tilde{u}(x,z)$*, then there exists a representation of the optimal solution such that* $\Gamma^*_t(y,r)>0$*,* $\mathbb{P}$*-a.s. for all* $t\in[0,T]$ *and* $\tilde{c}^*(x,z)$ *and* $\Gamma^*(y,r)$ *are related by:*

$$\begin{aligned}&\Gamma^*_t(y,r)=U'(t,\tilde{c}^*_t(x,z))\quad\text{or}\quad \tilde{c}^*_t(x,z)=I(t,\Gamma^*_t(y,r)),\\ &\left\langle\Gamma^*(y,r),\tilde{c}^*(x,z)\right\rangle=xy-zr.\end{aligned} \tag{2.4.8}$$

(iv) If we restrict the choice of initial wealth x *and initial standard of living* z *such that* $(x,z)\in\bar{\mathcal{H}}\subset\mathcal{H}$*, the solution* $c^*_t(x,z)$ *to our primal utility optimization problem* (2.3.10) *exists and is unique, moreover,*

$$\tilde{c}^*_t(x,z)=c^*_t(x,z)-Z^*_t(x,z).$$

2.5 Proofs of Main Results

2.5.1 The Proof of Theorem 2.4.2

The following Proposition will serve as the key step to build some future Bipolar relationships:

Proposition 2.5.1. *Assume all assumptions of Theorem 2.4.2 hold true. Then the families* $\left(\widetilde{\mathcal{A}}(x,z)\right)_{(x,z)\in\mathcal{H}}$ *and* $\left(\widetilde{\mathcal{Y}}(y,r)\right)_{(y,r)\in\mathcal{R}}$ *have the following properties:*

(i) For any $(x,z)\in\mathcal{H}$*, the set* $\widetilde{\mathcal{A}}(x,z)$ *contains a strictly positive random variable on the product space. A nonnegative random variable* $\tilde{c}$ *belongs to* $\widetilde{\mathcal{A}}(x,z)$ *if and only if*

$$\left\langle \tilde{c},\Gamma\right\rangle \leq xy-zr \quad \textit{for all } (y,r)\in\mathcal{R} \textit{ and } \Gamma\in\widetilde{\mathcal{Y}}(y,r). \tag{2.5.1}$$

(ii) For any $(y,r)\in\mathcal{R}$*, the set* $\widetilde{\mathcal{Y}}(y,r)$ *contains a strictly positive random variable on the product space. A nonnegative random variable* Γ *belongs to* $\widetilde{\mathcal{Y}}(y,r)$ *if and only if*

$$\left\langle \tilde{c},\Gamma\right\rangle \leq xy-zr \textit{ for all } (x,z)\in\mathcal{H} \textit{ and } \tilde{c}\in\widetilde{\mathcal{A}}(x,z). \tag{2.5.2}$$

In order to prove Proposition 2.5.1, for any $p>0$, we denote by $\mathcal{M}(p)$ the subset of $\mathcal{M}$ that consists of measure densities $Y\in\mathcal{M}$ such that $\left\langle w,Y\right\rangle = p$. Then for any density process $Y\in\mathcal{M}(p)$, define the auxiliary set as

$$\widetilde{\mathcal{M}}(p)\triangleq\left\{\Gamma\in\mathbb{L}^0_+:\Gamma_t=Y_t+\delta_t\mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)dv}Y_s ds\Big|\mathcal{F}_t\Big],\ \forall t\in[0,T],\ Y\in\mathcal{M}(p)\right\}. \tag{2.5.3}$$

We have $\left\langle \tilde{w},\Gamma\right\rangle = \left\langle w,Y\right\rangle = p$.

Define $\mathcal{P}$ as the open interval $\mathcal{P}=(\underline{p},\bar{p})$, where $\underline{p},\bar{p}$ are defined in (2.3.19) and (2.3.20). We have the following result.

Lemma 2.5.2. *Assume that conditions of Proposition* 2.5.1 *hold true and let* $p > 0$*. Then the set* $\widetilde{\mathcal{M}}(p)$ *is not empty if and only if* $p \in \mathcal{P} = (\underline{p}, \bar{p})$*, where* $\underline{p}, \bar{p}$ *are defined in* (2.3.19) *and* (2.3.20) *. In particular,*

$$\bigcup_{p \in \mathcal{P}} \widetilde{\mathcal{M}}(p) = \widetilde{\mathcal{M}}. \tag{2.5.4}$$

where the set $\widetilde{\mathcal{M}}$ *is defined by* (2.3.13)*.*

Proof. The proof reduces to verifying that $\mathcal{P} = \mathcal{P}'$, where we define

$$\mathcal{P}' \triangleq \{p > 0 : \ \widetilde{\mathcal{M}}(p) \neq \emptyset\}.$$

Similar to the proof of Lemma 8 of *Hugonnier and Kramkov* [35], one direction inclusion that $\mathcal{P} \subseteq \mathcal{P}'$ is obvious.

For the inverse, let $p \in \mathcal{P}'$, $(x, z) \in cl\mathcal{H}$, $\Gamma \in \widetilde{\mathcal{M}}(p)$, and we first claim there exists a $\tilde{c} \in \widetilde{\mathcal{A}}(x, z)$ such that

$$\bar{\mathbb{P}}[\tilde{c} \succ 0] > 0,$$

so we get

$$0 < \left\langle \tilde{c}, \Gamma \right\rangle \leq x - zp.$$

As (x, z) is an arbitrary element of $cl\mathcal{H}$, we have $p \in \mathcal{P}$.

As for the above claim, according to Theorem 2.11 of *Schachermayer* [69], Assumption (2.4.2) guarantees that for all $Y \in \mathcal{M}$, we have

$$\underline{p} < \langle w, Y \rangle < \bar{p},$$

which is

$$\underline{p} < \langle \tilde{w}, \Gamma \rangle < \bar{p},$$

for all the $\Gamma \in \widetilde{\mathcal{M}}$. Then by the definition of $cl\mathcal{H}$ in Lemma 2.3.5, we observe that for any $(x, z) \in cl\mathcal{H}$, we will have

$$x - z\big\langle \tilde{w}, \Gamma \big\rangle > 0,$$

for all the $\Gamma \in \widetilde{\mathcal{M}}$, and the claim holds by the definition of $\widetilde{\mathcal{A}}(x, z)$. $\square$

Lemma 2.5.3. *Assume that conditions of Proposition* 2.5.1 *hold true and let* $p \in \mathcal{P} = (\underline{p}, \bar{p})$*, we have then* $\widetilde{\mathcal{M}}(p) \subseteq \widetilde{\mathcal{Y}}(1, p)$*.*

Proof. The conclusion can be directly derived in light of the definition of $\widetilde{\mathcal{A}}(x, z)$ and $\widetilde{\mathcal{Y}}(1, p)$. $\square$

Lemma 2.5.4. *Assume that conditions of Proposition* 2.5.1 *hold true. For any* $(x, z) \in \mathcal{H}$*, a nonnegative random variable* $\tilde{c}$ *belongs to* $\tilde{\mathcal{A}}(x, z)$ *if and only if*

$$\big\langle \tilde{c}, \Gamma \big\rangle \leq x - zp \;\; \textit{for all } p \in \mathcal{P} \textit{ and } \Gamma \in \widetilde{\mathcal{M}}(p). \tag{2.5.5}$$

Proof. If $\tilde{c} \in \widetilde{\mathcal{A}}(x, z)$, the definition of $\widetilde{\mathcal{A}}(x, z)$ and the fact $\widetilde{\mathcal{M}}(p) \subset \widetilde{\mathcal{M}}$ guarantee the validity of (2.5.5).

On the other hand, for any $\tilde{c} \in \mathbb{L}^0_+$ such that (2.5.5) holds true, we will have:

$$\begin{aligned}
\sup_{\Gamma \in \widetilde{\mathcal{M}}} \big\langle \tilde{c} + z\tilde{w}, \Gamma \big\rangle &= \sup_{p \in \mathcal{P}} \sup_{\Gamma \in \widetilde{\mathcal{M}}(p)} \big\langle \tilde{c} + z\tilde{w}, \Gamma \big\rangle \\
&= \sup_{p \in \mathcal{P}} \sup_{\Gamma \in \widetilde{\mathcal{M}}(p)} \Big(\big\langle \tilde{c}, \Gamma \big\rangle + zp \Big) \leq x.
\end{aligned}$$

The claim holds according to the definition of $\widetilde{\mathcal{A}}(x,z)$. $\square$

PROOF OF PROPOSITION 2.5.1.

For the validity of assertion (i), consider $(x,z) \in \mathcal{H}$, there exists a $\lambda > 0$ such that $(x-\lambda, z) \in \mathcal{H}$ since $\mathcal{H}$ is an open set.

Let $\tilde{c} \in \widetilde{\mathcal{A}}(x-\lambda, z)$, we will have for any $\Gamma \in \widetilde{\mathcal{M}}$, and $\tilde{w}_t = e^{-\int_0^t \alpha_v dv} \succ 0$,

$$\big\langle \tilde{c}, \Gamma \big\rangle \leq x - \lambda - z \big\langle \tilde{w}, \Gamma \big\rangle. \tag{2.5.6}$$

By Assumption (2.3.3) and Proposition 2.3.3, we define $\rho_t \triangleq \frac{\lambda}{\bar{p}} \tilde{w}_t > 0$ for all $t \in [0,T]$, then for all $\Gamma \in \widetilde{\mathcal{M}}$:

$$\begin{aligned}\big\langle \rho, \Gamma \big\rangle \leq \big\langle \tilde{c} + \rho, \Gamma \big\rangle &\leq x - \lambda - z\big\langle \tilde{w}, \Gamma \big\rangle + \frac{\lambda}{\bar{p}} \big\langle \tilde{w}, \Gamma \big\rangle \\ &\leq x - \lambda - z\big\langle \tilde{w}, \Gamma \big\rangle + \lambda \leq x - z \big\langle \tilde{w}, \Gamma \big\rangle.\end{aligned}$$

Hence, we have shown the existence of a strictly positive element $\rho_t \succ 0 \in \widetilde{\mathcal{A}}(x,z)$ by the definition of $\widetilde{\mathcal{A}}(x,z)$.

If (2.5.1) holds for some $\tilde{c} \in \mathbb{L}^0_+$. The density process $\Gamma \in \widetilde{\mathcal{M}}(p)$ belongs to $\widetilde{\mathcal{Y}}(1,p)$ for all $p \in \mathcal{P}$ by Lemma 2.5.3, and hence (2.5.5) holds. Lemma 2.5.4 then implies that $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$.Conversely, suppose now $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$, the definition of sets $\widetilde{\mathcal{Y}}(y,r)$, $(y,r) \in \mathcal{R}$ implies (2.5.1) and we complete the proof of assertion (i).

For the proof the assertion (ii), notice

$$k\widetilde{\mathcal{Y}}(y,r) = \widetilde{\mathcal{Y}}(ky,kr) \ \text{ for all } k > 0, \ (y,r) \in \mathcal{R}.$$

therefore we just need to consider $(y,r) = (1,p)$ for some $p \in \mathcal{P}$. Lemma 2.5.3 implies $\Gamma \in \widetilde{\mathcal{M}}(p) \subseteq \widetilde{\mathcal{Y}}(1,p)$, and the existence of $Y \succ 0 \in \mathcal{M}(p)$ takes care of the existence $\Gamma \succ 0 \in \widetilde{\mathcal{M}}(p)$, $\bar{\mathbb{P}}$-a.s.

The second part is a direct consequence of the definition of $\widetilde{\mathcal{Y}}(y, r)$. $\square$

For the proof of Theorem 2.4.2, we will also need the following lemmas:

Lemma 2.5.5. *Under assumptions of Theorem* 2.4.2*, the value function* $\tilde{u}$ *is* $(-\infty, \infty)$*-valued on* $\mathcal{H}$*.*

Proof. First, by Lemma 2.2.2, the assumption $AE_0[U] < \infty$ implies that for any positive constant $s > 0$, the existence of $s_1 > 0$ and $s_2 > 0$ such that for all $t \in [0, T]$:

$$U(t, x/s) \geq s_1 U(t, x) + s_2, \quad x > 0, \tag{2.5.7}$$

According to Assumption (2.3.4) and the proof of Proposition 2.5.1, for each fixed pair $(x, z) \in \mathcal{H}$, there exists $\lambda = \lambda(x, z) > 0$ such that $\frac{\lambda}{\bar{p}}\tilde{w}_t \in \widetilde{\mathcal{A}}(x, z)$, therefore we deduce that $\bar{x}\tilde{w}_t \in \widetilde{\mathcal{A}}(\frac{\bar{x}\bar{p}}{\lambda}x, \frac{\bar{x}\bar{p}}{\lambda}z)$, and

$$\tilde{u}(\frac{\bar{x}\bar{p}}{\lambda}x, \frac{\bar{x}\bar{p}}{\lambda}z) = \sup_{\tilde{c} \in \widetilde{\mathcal{A}}(\frac{\bar{x}\bar{p}}{\lambda}x, \frac{\bar{x}\bar{p}}{\lambda}z)} \mathbb{E}\Big[\int_0^T U(t, \tilde{c}_t)dt\Big] \geq \mathbb{E}\Big[\int_0^T U(t, \bar{x}\tilde{w}_t)dt\Big] > -\infty,$$

hence, for any $(x, z) \in \mathcal{H}$, we get the existence of a constant $s(x, z) > 0$, such that $\tilde{u}(sx, sz) > -\infty$, with $s(x, z) = \frac{\bar{x}\bar{p}}{\lambda}$.

Since, for any constant $s > 0$,

$$\widetilde{\mathcal{A}}(x, z) = \widetilde{\mathcal{A}}(sx, sz)/s,$$

we derive $\tilde{u}(x, z) > -\infty$ if $\tilde{u}(sx, sz) > -\infty$ holds for a constant $s = s(x, z) > 0$, follow the result above, we conclude that $\tilde{u}(x, z) > -\infty$ in the whole domain

$\mathcal{H}$.

Now, since the set $\mathcal{H}$ is open and $\tilde{u}(x, z) < \infty$ for some $(x, z) \in \mathcal{H}$ by assumption (2.4.6), we deduce that $\tilde{u}$ is finitely valued on $\mathcal{H}$ by the concavity of $\tilde{u}$ on $\mathcal{H}$. And the proof is complete. $\square$

Before we state the next lemma, let's introduce a special concept of compactness which was originally defined in *Žitković* [78].

Definition 2.5.1. A convex subset C of a topological vector space X is said to be *convexly compact* if for any non-empty set A and any family $\{F_a\}_{a \in A}$ of closed, convex subsets of C, the condition

$$\forall D \in Fin(A), \bigcap_{a \in D} F_a \neq \emptyset \implies \bigcap_{a \in A} F_a \neq \emptyset$$

where the set $Fin(A)$ consists of all non-empty finite subsets of A for an arbitrary non-empty set A.

Without the restriction that the sets $\{F_a\}_{a \in A}$ must be convex, this definition would be equivalent to compactness in the original sense. Thus any convex and compact set is convexly compact and Definition 2.5.1 extends the concept of compactness.

Žitković [78] furthermore derived an easy characterization on the space of non-negative, measurable functions, see Theorem 3.1 of *Žitković* [78] which states that

Theorem 2.5.6. *A closed and convex subset C of $\mathbb{L}^0_+$ is convexly compact if and only if it is bounded in finite measure.*

Based on the above theorem, we have the following lemma on the convex compactness of sets $\widetilde{\mathcal{A}}(x,z)$ and $\widetilde{\mathcal{Y}}(y,r)$:

Lemma 2.5.7. *For each pair $(x,z) \in \mathcal{H}$ and $(y,r) \in \mathcal{R}$, the sets $\widetilde{\mathcal{A}}(x,z)$ and $\widetilde{\mathcal{Y}}(y,r)$ are convex, solid and closed in the topology of convergence in measure $\bar{\mathbb{P}}$. Moreover, they are both bounded in $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \bar{\mathbb{P}})$, hence they are both convexly compact.*

Proof. For $(y,r) \in \mathcal{R}$, we now define two auxiliary sets as

$$\begin{aligned} \mathfrak{H}(y,r) &\triangleq \Big\{(x,z) \in \mathcal{H} : \ xy - zr \leq 1\Big\} \\ \mathfrak{A}(k) &\triangleq \bigcup_{(x,z)\in k\mathfrak{H}(y,r)} \widetilde{\mathcal{A}}(x,z), \end{aligned} \tag{2.5.8}$$

and denote by $\widetilde{\mathfrak{A}}(k)$ the closure of $\mathfrak{A}(k)$ with respect to convergence in measure $\bar{\mathbb{P}}$.

From Proposition 2.5.1, we deduce that

$$\Gamma \in \widetilde{\mathcal{Y}}(y,r) \ \Leftrightarrow \ \big\langle \tilde{c}, \Gamma \big\rangle \leq 1, \quad \forall \tilde{c} \in \widetilde{\mathfrak{A}}(1)$$

Hence, sets $\widetilde{\mathcal{Y}}(y,r)$ and $\widetilde{\mathfrak{A}}(1)$ satisfy

$$\widetilde{\mathcal{Y}}(y,r) = \widetilde{\mathfrak{A}}(1)^{\circ}.$$

At the same time, by its definition, we have $\widetilde{\mathfrak{A}}(1)$ itself is closed, convex and solid, by the Bipolar theorem in *Brannath and Schachermayer* [11], we have $\widetilde{\mathfrak{A}}(1) = \widetilde{\mathfrak{A}}(1)^{\circ\circ}$, and hence we have the following Bipolar relationship:

$$\begin{aligned} \widetilde{\mathfrak{A}}(1) &= \widetilde{\mathcal{Y}}(y,r)^{\circ} \\ \widetilde{\mathcal{Y}}(y,r) &= \widetilde{\mathfrak{A}}(1)^{\circ}. \end{aligned} \tag{2.5.9}$$

The Bipolar theorem on $\mathbb{L}^0_+$ gives the convexity, solidness and closure in measure $\bar{\mathbb{P}}$.

Similarly, for $(x, z) \in \mathcal{H}$, now define the set:

$$\begin{aligned} \mathfrak{R}(x,z) &\triangleq \{(y,r) \in \mathcal{R} : xy - zr \leq 1\}, \\ \mathfrak{Y}(k) &\triangleq \bigcup_{(y,r)\in k\mathfrak{R}(x,z)} \widetilde{\mathcal{Y}}(y,r), \end{aligned} \tag{2.5.10}$$

and denote by $\widetilde{\mathfrak{Y}}(k)$ the closure of $\mathfrak{Y}(k)$ with respect to convergence in measure $\bar{\mathbb{P}}$.

Now, again Proposition 2.5.1 implies

$$\tilde{c} \in \widetilde{\mathcal{A}}(x,z) \Leftrightarrow \left\langle \tilde{c}, \Gamma \right\rangle \leq 1, \ \forall \Gamma \in \widetilde{\mathfrak{Y}}$$

and the Bipolar relationship:

$$\begin{aligned} \widetilde{\mathfrak{Y}}(1) &= \widetilde{\mathcal{A}}(x,z)^{\circ} \\ \widetilde{\mathcal{A}}(x,z) &= \widetilde{\mathfrak{Y}}(1)^{\circ}. \end{aligned} \tag{2.5.11}$$

Hence, we also have $\widetilde{\mathcal{A}}(x,z)$ is convex, solid and closed in the topology of convergence in measure $\bar{\mathbb{P}}$.

Moreover, thanks to the existence of $0 \prec \Gamma \in \widetilde{\mathcal{M}}(p)$ which is also in $\widetilde{\mathcal{Y}}(1,p)$, we deduce the set $\widetilde{\mathcal{A}}(x,z)$ is bounded in measure $\bar{\mathbb{P}}$ by Proposition 2.5.1 part (i).

Similarly, as in the proof of Proposition 2.5.1, we have derived the existence of $\lambda = \lambda(x,z)$ such that $0 \prec \rho_t = \frac{\lambda}{\bar{p}} \tilde{w}_t \in \widetilde{\mathcal{A}}(x,z)$, due to Proposition 2.5.1 part (ii), we get the set $\widetilde{\mathcal{Y}}(y,r)$ is also bounded in measure $\bar{\mathbb{P}}$. And therefore both of them are convexly compact in $\mathbb{L}^0_+$. $\square$

A major difficulty arises in the proof of the existence of the dual optimizer in our setting due to the lack of integrability of the dual process $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$ for $(y,r) \in \mathcal{R}$. In fact, the trick of applying de la Vallée-Poussin theorem in the proof of Lemma 3.2 in *Kramkov and Schachermayer* [48] and Lemma A.1 in *Karatzas and Žitković* [43] does not work. And the argument of contradiction mimicking the proof of Lemma 1 in *Kramkov and Schachermayer* [49] using the subsequence splitting lemma will also fail by observing the constant may not be contained in the dual space. Contrary to the results in the literature, much effort has to be made to modify the classic analysis, where the Assumptions of $AE[U]_0 < \infty$ and $\mathbb{E}\Big[\int_0^T U(t, \bar{x}\tilde{w}_t)dt\Big] > -\infty$ are critical for the procedure of our proof of the following lemmas.

Lemma 2.5.8. *Under assumptions of theorem* 2.4.2*, we have for each fixed* $(y,r) \in \mathcal{R}$

$$\sup_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T V^-(t,\Gamma_t)dt\Big] < \infty.$$

Proof. Assumption (2.3.4) admits the existence of $\bar{x}\tilde{w}_t \in \mathbb{L}^0_+$ such that $\mathbb{E}\Big[\int_0^T U(t, \bar{x}\tilde{w}_t)dt\Big] > -\infty$, and moreover, by the proof of Proposition 2.5.1, we also know for each fixed $(y,r) \in \mathcal{R}$, find the fixed pair $(x,z) \in \widetilde{\mathfrak{H}}(y,r)$, there exists a constant $\lambda(x,z) > 0$ such that $\tilde{w} \in \widetilde{\mathfrak{A}}(\frac{\bar{p}}{\lambda})$, where $\bar{p}$ is defined by (2.3.19). Taking into account the inequality $U(t,x) \leq V(t,y) + xy$, we have for any $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$ and $y_0(t) \triangleq \inf\{y>0: \ V(t,y)<0\}$

$$\mathbb{E}\Big[\int_0^T V^-(t,\Gamma_t)dt\Big] \leq -\mathbb{E}\Big[\int_0^T V(t,\Gamma_t 1_{\{\Gamma_t\geq y_0(t)\}} + y_0(t)1_{\{\Gamma_t<y_0(t)\}})dt\Big]$$

$$\leq -\mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big] + \bar{x}\mathbb{E}\Big[\int_0^T \tilde{w}_t\Gamma_t dt\Big] + \bar{x}\mathbb{E}\Big[\int_0^T \tilde{w}_t(y_0(t)-\Gamma_t)1_{\{\Gamma_t<y_0(t)\}}dt\Big]$$

$$\leq -\mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big] + \bar{x}\frac{\bar{p}}{\lambda} + \bar{x}\int_0^T y_0(t)dt,$$

which is finitely valued and independent of the initial choice of Γ since we have $\tilde{w}_t \triangleq e^{\int_0^t(-\alpha_v)dv} \leq 1$ for $t\in[0,T]$ and $\sup_{t\in[0,T]} y_0(t) < \infty$ by Assumption (2.2.12), and thus our conclusion holds true. □

Lemma 2.5.9. *Under assumptions of Theorem* 2.4.2*, we have for any* $(y,r)\in\mathcal{R}$*,* $\Big(V^-(\cdot,\Gamma_\cdot)\Big)$ *is uniformly integrable for all* $\Gamma\in\widetilde{\mathcal{Y}}(y,r)$*.*

Proof. By Corollary 2.2.3, the assumption $AE_0[U]<\infty$ is equivalent to the following assertions:

$$\exists y_0>0, \text{ and } \mu\in(1,2), \quad \forall y\geq y_0, \quad V(t,2y)\geq \mu V(t,y). \tag{2.5.12}$$

Let $y_0>0$ and $\mu\in(1,2)$ be the constants in the above (2.5.12). Take $\gamma=\log_2\mu\in(0,1)$, we define the auxiliary function $\widetilde{V}(t,y):[0,\infty)\times(0,\infty)\to\mathbb{R}$ by

$$\widetilde{V}(t,y)\triangleq\begin{cases} -\frac{2y_0}{\gamma}V'(t,2y_0)-V(t,y), & y\geq 2y_0,\\ -V(t,2y_0)-\frac{2y_0}{\gamma}V'(t,2y_0)(\frac{y}{2y_0})^\gamma, & y<2y_0.\end{cases} \tag{2.5.13}$$

For each fixed $t>0$, $\widetilde{V}(t,y)$ is a nonnegative, concave, and nondecreasing function which agrees with $-V(t,y)$ up to a constant for large enough values of y and satisfies

$$\widetilde{V}(t,2y)\leq\mu\widetilde{V}(t,y), \text{ for all } y>0. \tag{2.5.14}$$

Lemma 2.5.8 asserts

$$\sup_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T V^-(t,\Gamma_t)dt\Big] < \infty,$$

and hence in light of the fact that V^- and $\widetilde{V}$ differ only by a constant in a neighborhood of ∞, we will get

$$\sup_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \widetilde{V}(t,\Gamma_t)dt\Big] < \infty. \tag{2.5.15}$$

The validity of uniform integrability of the sequence $\Big(V^-(\cdot,\Gamma^n_\cdot)\Big)_{n\geq 1}$ for $\Gamma^n \in \widetilde{\mathcal{Y}}(y,r)$, is therefore equivalent to the uniform integrability of $(\widetilde{V}(\cdot,\Gamma^n_\cdot))_{n\geq 1}$.

To this end, we argue by contradiction. Suppose this sequence is not uniformly integrable, then by Rosenthal's subsequence splitting lemma, we can find a subsequence $(f^n)_{n\geq 1}$, a constant $\varepsilon > 0$ and a disjoint sequence $(A^n)_{n\geq 1}$ of $(\Omega \times [0,T], \mathcal{O})$ with

$$A^n \in \mathcal{O}, \quad A^i \cap A^j = \varnothing \quad \text{if } i \neq j,$$

such that

$$\mathbb{E}\Big[\int_0^T \widetilde{V}(t, f^n_t)1_{A^n}dt\Big] \geq \varepsilon, \quad \text{for } n \geq 1$$

We define the sequence of random variables $(h^n)_{n\geq 1}$

$$h^n_t = \sum_{k=1}^n f^k_t 1_{A^k}.$$

For any $\tilde{c} \in \widetilde{\mathfrak{A}}(1)$,

$$\big\langle \tilde{c}, h^n \big\rangle \leq \sum_{k=1}^n \big\langle \tilde{c}, f^k \big\rangle \leq n.$$

Hence $\frac{h^n}{n} \in \widetilde{\mathcal{Y}}(y,r)$.

One the other hand,

$$\mathbb{E}\Big[\int_0^T \widetilde{V}(t, h_t^n)dt\Big] \geq \sum_{k=1}^{n} \mathbb{E}\Big[\int_0^T \widetilde{V}(t, f_t^k)1_{A^k}dt\Big] \geq \varepsilon n,$$

and therefore by taking $n = 2^m$, via iteration, it produces

$$\begin{aligned}\mu^m \sup_{\Gamma_t \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \widetilde{V}(t, \Gamma_t)dt\Big] &\geq \mu^m \mathbb{E}\Big[\int_0^T \widetilde{V}(t, \frac{h_t^{2^m}}{2^m})dt\Big] \\ &\geq \mathbb{E}\Big[\int_0^T \widetilde{V}(t, h_t^{2^m})dt\Big] \geq 2^m \epsilon,\end{aligned}$$

since $\mu \in (1,2)$, this contradicts (2.5.15) for m large enough, therefore the conclusion holds true. $\square$

Lemma 2.5.10. *For any pair $(y,r) \in \mathcal{R}$ such that $\tilde{v}(y,r) < \infty$, the optimal solution Γ^* to the optimization problem* (2.4.5) *exists and is unique.*

Proof. Now fix $(y,r) \in \mathcal{R}$, let $(\Gamma^n)_{n\geq 1}$ be a sequence in $\widetilde{\mathcal{Y}}(y,r)$ such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T V(t, \Gamma_t^n)dt\Big] = \tilde{v}(y,r).$$

There exists a sequence of forward convex combinations $f^n \in$ $conv(\Gamma^n, \Gamma^{n+1}, \ldots)$ which converges almost surely to a random variable Γ^* with values in $[0,\infty]$. Since the set $\widetilde{\mathcal{Y}}(y,r)$ is closed and bounded in measure $\bar{\mathbb{P}}$ in $\mathbb{L}_+^0$ by Lemma 2.5.7, we deduce that Γ^* is almost surely finitely valued, moreover, Γ^* belongs to $\widetilde{\mathcal{Y}}(y,r)$. We claim that Γ^* is the optimal solution to (2.4.5), that is

$$\mathbb{E}\Big[\int_0^T V(t, \Gamma_t^*)dt\Big] = \tilde{v}(y,r).$$

The concavity of V produces

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^T V(t,f_t^n)dt\Big]\leq \tilde{v}(y,r),$$

and Fatou's lemma implies

$$\liminf_{n\to\infty}\mathbb{E}\Big[\int_0^T V^+(t,f_t^n)dt\Big]\geq \mathbb{E}\Big[\int_0^T V^+(t,\Gamma_t^*)dt\Big].$$

The optimality of Γ_t^* will follow if we can show

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^T V^-(t,f_t^n)dt\Big]= \mathbb{E}\Big[\int_0^T V^-(t,\Gamma_t^*)dt\Big], \tag{2.5.16}$$

but the validity of (2.5.16) is a consequence of Lemma 2.5.9. □

For the proof of conjugate duality relations between value functions $\tilde{u}(x,z)$ and $\tilde{v}(y,r)$, similar to Lemma 11 of *Hugonnier and Kramkov* [35], we have the following general result:

Lemma 2.5.11. *If $\mathcal{G}\subseteq\mathbb{L}_+^0$ is convex and contains a strictly positive random variable. Then*

$$\sup_{g\in\mathcal{G}}\mathbb{E}\Big[\int_0^T U(t,xg_t)dt\Big]=\sup_{g\in cl\mathcal{G}}\mathbb{E}\Big[\int_0^T U(t,xg_t)dt\Big],\quad x>0$$

where $cl\mathcal{G}$ denotes the closure of $\mathcal{G}$ with respect to convergence in measure $\bar{\mathbb{P}}$.

Proof. Without loss of generality, we can assume that the set $\mathcal{G}$ contains an element $\rho_t>0$. Denote, for $x>0$,

$$\phi(x)\triangleq\sup_{g\in\mathcal{G}}\mathbb{E}\Big[\int_0^T U(t,xg_t)dt\Big],\quad \psi(x)\triangleq\sup_{g\in cl\mathcal{G}}\mathbb{E}\Big[\int_0^T U(t,xg_t)dt\Big]$$

Clearly, ϕ and ψ are concave functions and $\phi \leq \psi$. If $\phi(x) = \infty$ for some $x > 0$, then, due to concavity, ϕ is infinite for all arguments and the assertion of the lemma is trivial. Hereafter we assume that ϕ is finite.

Fix $x > 0$ and $g \in cl\mathcal{G}$. Let $(g^n)_{n\geq 1}$ be a sequence in $\mathcal{G}$ that converges to g almost surely in measure $\bar{\mathbb{P}}$. For any $\delta > 0$, we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^T U(t, xg_t)dt\Big] &\leq \mathbb{E}\Big[\int_0^T U(t, xg_t + \delta\rho_t)dt\Big] \\ &\leq \liminf_{n\to\infty} \mathbb{E}\Big[\int_0^T U(t, xg_t^n + \delta\rho_t)dt\Big] \leq \phi(x+\delta),\end{aligned}$$

where the first inequality holds true because U is increasing in x, the second one follows from Fatou's lemma and the third one follows from the facts that $\mathcal{G}$ is convex and contains $\rho_t > 0$. Since ϕ is concave, it is continuous. It follows that

$$\psi(x) = \sup_{g\in cl\mathcal{E}} \mathbb{E}\Big[\int_0^T U(t, xg_t)dt\Big] \leq \lim_{\delta\to 0}\phi(x+\delta) = \phi(x).$$

□

Lemma 2.5.12. *For* $\tilde{w}_t \triangleq e^{\int_0^t(-\alpha_v)dv}$*, we have the following result:*

$$\mathbb{E}\Big[\int_0^T V^-(t, U'(t, \tilde{w}_t))dt\Big] < \infty. \tag{2.5.17}$$

Proof. Similar to the proof of Lemma 2.5.8, recall the Assumption that $\mathbb{E}\Big[\int_0^T U(t, \bar{x}\tilde{w}_t)dt\Big] > -\infty$, taking into account the inequality $U(t, x) < V(t, y) +$

xy, we have for any $y_0(t) \triangleq \inf\{y > 0 : \ V(t, y) < 0\}$,

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T V^-(t, U'(t, \tilde{w}_t))dt\Big] \\
&\leq -\mathbb{E}\Big[\int_0^T V(t, U'(t, \tilde{w}_t)1_{\{U'(t,\tilde{w}_t)\geq y_0(t)\}} + y_0(t)1_{\{U'(t,\tilde{w}_t)<y_0(t)\}})dt\Big] \\
&\leq -\mathbb{E}\Big[\int_0^T U(t, \bar{x}\tilde{w}_t)dt\Big] + \bar{x}\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)dt\Big] \qquad (2.5.18) \\
&+\bar{x}\mathbb{E}\Big[\int_0^T \tilde{w}_t(y_0(t) - U'(t, \tilde{w}_t))1_{\{U'(t,\tilde{w}_t)<y_0(t)\}}dt\Big] \\
&\leq -\mathbb{E}\Big[\int_0^T U(t, \bar{x}\tilde{w}_t)dt\Big] + \bar{x}\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)dt\Big] + \bar{x}\int_0^T y_0(t)dt.
\end{aligned}$$

We already know the first term and the third term are bounded, as for the second term, we have two different cases:

1. If we have $\bar{x} \leq 1$, then we can rewrite the second term as

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)dt\Big] &= \mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)1_{\{\tilde{w}_t\leq x_0\}}dt\Big] \\
&+ \mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)1_{\{\tilde{w}_t> x_0\}}dt\Big],
\end{aligned}$$

where x_0 is the uniform constant in Corollary 2.2.3 such that for all $t \in [0, T]$,

$$x U'(t, x) < \Big(\frac{\gamma}{1-\gamma}\Big)\Big(-U(t, x)\Big) \ \text{ for } 0 < x \leq x_0. \qquad (2.5.19)$$

Again, use the fact that $\tilde{w} \preceq 1$, we have

$$\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)1_{\{\tilde{w}_t> x_0\}}dt\Big] < \infty,$$

and we also have

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)1_{\{\tilde{w}_t\leq x_0\}}dt\Big] &\leq -\Big(\frac{\gamma}{1-\gamma}\Big)\mathbb{E}\Big[\int_0^T U(t, \tilde{w}_t)dt\Big] \\
&\leq -\Big(\frac{\gamma}{1-\gamma}\Big)\mathbb{E}\Big[\int_0^T U(t, \bar{x}\tilde{w}_t)dt\Big] < \infty,
\end{aligned}$$

by using the inequality (2.5.19), the increasing property of $U(t,x)$ with respect to x and the Assumption (2.3.4).

2. If we have $\bar{x} > 1$, then we rewrite the second term as:

$$\begin{aligned}\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)dt\Big] &= \mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t \le x_0\}}dt\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t > x_0\}}dt\Big],\end{aligned}$$

where x_0 is the uniform constant in Corollary 2.2.3 such that for all $t \in [0,T]$, the inequality (2.5.19) holds and moreover,

$$U(t, \frac{1}{\bar{x}}x) > (\frac{1}{\bar{x}})^{-\frac{\gamma}{1-\gamma}} U(t,x) \quad \text{for } 0 < x \le x_0, \tag{2.5.20}$$

holds for all $t \in [0,T]$.

Then, again, the second term is bounded since $\bar{x}\tilde{w} \preceq \bar{x}$, and for the first term, we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t \le x_0\}}dt\Big] &\le -\Big(\frac{\gamma}{1-\gamma}\Big)\mathbb{E}\Big[\int_0^T U(t,\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t \le x_0\}}dt\Big] \\ &\le -\Big(\frac{\gamma}{1-\gamma}\Big)(\frac{1}{\bar{x}})^{-\frac{\gamma}{1-\gamma}}\mathbb{E}\Big[\int_0^T U(t,\bar{x}\tilde{w}_t)dt\Big] < \infty,\end{aligned}$$

by the inequality (2.5.19) and (2.5.20) and the Assumption (2.3.4).

Hence we proved the second term in (2.5.18) is also finite, and we can therefore conclude that result (2.5.17) holds true. □

We should again emphasize the fact that the auxiliary dual domain $\widetilde{\mathcal{Y}}(y,r)$ is not necessary a subset of $\mathbb{L}^1$, and hence we have to revise the usual Minimax theorem based on $\mathbb{L}^1$ to derive the important conjugate duality relationship. Fortunately, the following Minimax theorem proved by *Kauppila* [44] can serve as a substitute tool on the space $\mathbb{L}^0_+$ without any priori assumption on the integrability of the dual process.

Theorem 2.5.13 (Minimax Theorem)**.** *Let A be a nonempty convex subset of a topological space, and B a nonempty, closed, convex, and convexly compact subset of a topological vector space. Let $H : A \times B \to \mathbb{R}$ be convex on A, and concave and upper-semicontinuous on B. Then*

$$\sup_B \inf_A H = \inf_A \sup_B H.$$

See the detail proof in Theorem A.1 in Appendix A by *Kauppila* [44]. We remark this Minimax Theorem is a relaxed version of Theorem 4.9 by *Žitković* [78]. Contrary to the assumption of *Žitković* [78] that the target functional needs to be semi-continuous with respect to both vector spaces, *Kauppila* [44] only requires the functional has semi-continuity property on one of the vector spaces, which can be applied to our case.

Lemma 2.5.14. *Under assumptions of Theorem* 2.4.2*, the conjugate duality relations hold:*

$$\begin{aligned} \tilde{u}(x,z) &= \inf_{(y,r)\in\mathcal{R}} \{\tilde{v}(y,r) + xy - zr\}, \qquad (x,z)\in\mathcal{H}, \\ \tilde{v}(y,r) &= \sup_{(x,z)\in\mathcal{H}} \{\tilde{u}(x,z) - xy + zr\}, \qquad (y,r)\in\mathcal{R}. \end{aligned} \tag{2.5.21}$$

Proof. For $n > 0$, we define $\mathcal{S}_n$ as a subset in $\mathbb{L}^0_+(\Omega\times[0,T],\mathcal{O},\bar{\mathbb{P}})$ as

$$\mathcal{S}_n = \{\tilde{c}\in\mathbb{L}^0_+ : 0 \preceq \tilde{c} \preceq n\tilde{w}\}.$$

It is clear that sets $\mathcal{S}_n$ are closed, convex, and bounded in probability, and hence convexly compact in $\mathbb{L}^0_+$.

We will first show that the functional

$$\tilde{c} \mapsto \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big]$$

is upper-semicontinuous on $\mathcal{S}_n$ in the topology of convergence in measure $\bar{\mathbb{P}}$, for all $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$ and $(y,r) \in \mathcal{R}$:

In fact, by passing if necessary to a subsequence denoted by $(\tilde{c}^m)_{m\geq 1}$ converges almost surely to $\tilde{c} \in \mathcal{S}_n$, Fatou's lemma implies both

$$\liminf_{m\to\infty}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}^m_t)^- dt\Big] \geq \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)^- dt\Big], \tag{2.5.22}$$

and

$$\liminf_{m\to\infty}\mathbb{E}\Big[\int_0^T \tilde{c}^m_t\Gamma_t dt\Big] \geq \mathbb{E}\Big[\int_0^T \tilde{c}_t\Gamma_t dt\Big]. \tag{2.5.23}$$

Moreover, on $\mathcal{S}_n$, it is clear that $\mathbb{E}\Big[\int_0^T U(t,\tilde{c}^m_t)^+ dt\Big]$ is uniformly integrable, and hence

$$\lim_{m\to\infty}\mathbb{E}\Big[\int_0^T U(t,\tilde{c}^m_t)^+ dt\Big] = \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)^+ dt\Big]. \tag{2.5.24}$$

Now, together with (2.5.22) and (2.5.23), we have

$$\limsup_{m\to\infty}\mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}^m_t) - \tilde{c}^m_t\Gamma_t\Big)dt\Big] \leq \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big].$$

Noting that, by Lemma 2.5.7, $\widetilde{\mathcal{Y}}(y,r)$ is a closed convex subset of $\mathbb{L}^0_+$, we may use the above Minimax Theorem 2.5.13 to get the following equality, for n fixed:

$$\sup_{\tilde{c}\in\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big] = \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\sup_{\tilde{c}\in\mathcal{S}_n}\mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big].$$

Recall now the Bipolar relationship (2.5.9), and from the definition, we have

$$\bigcup_{(x,z)\in\mathcal{H}} \widetilde{\mathcal{A}}(x,z) = \bigcup_{k>0} \widetilde{\mathfrak{A}}(k). \tag{2.5.25}$$

As a preparation of the following proof, we define the auxiliary set

$$\mathfrak{A}'(k) \triangleq \Big\{ \tilde{c} \in \widetilde{\mathfrak{A}}(k) : \sup_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \langle \tilde{c}, \Gamma \rangle = k \Big\}$$

and clearly, we also have

$$\bigcup_{k>0} \widetilde{\mathfrak{A}}(k) = \bigcup_{(x,z)\in\mathcal{H}} \widetilde{\mathcal{A}}(x,z) = \bigcup_{k>0} \mathfrak{A}'(k). \tag{2.5.26}$$

We show first that

$$\begin{aligned} &\lim_{n\to\infty} \sup_{\tilde{c}\in\mathcal{S}_n} \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big] \\ &= \sup_{k>0} \sup_{\tilde{c}\in\mathfrak{A}'(k)} \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big]. \end{aligned} \tag{2.5.27}$$

The direction of inequality "$\geq$" holds by

$$\begin{aligned} &\lim_{n\to\infty} \sup_{\tilde{c}\in\mathcal{S}_n} \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big] \\ \geq &\lim_{n\to\infty} \sup_{\tilde{c}\in\mathfrak{A}'(k)\cap\mathcal{S}_n} \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big] \\ = &\sup_{\tilde{c}\in\mathfrak{A}'(k)} \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t) - \tilde{c}_t\Gamma_t\Big)dt\Big], \quad \forall k > 0, \end{aligned}$$

while the other direction "$\leq$" is obvious since for any $(x,z) \in \mathcal{H}$, we have $n\tilde{w} \in \mathfrak{A}'(n\bar{p})$, and hence $\mathcal{S}_n \subset \mathfrak{A}'(n\bar{p})$.

To show the next step, we need to prepare some finiteness results as below:

From definitions in Lemma 2.5.7 and by Lemma 2.5.11, we know

$$\sup_{\tilde{c}\in\widetilde{\mathfrak{A}}(k)} \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big] = \sup_{\tilde{c}\in\mathfrak{A}(k)} \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big] = \sup_{(x,z)\in k\mathfrak{H}(y,r)} \tilde{u}(x,z), \quad k>0, \tag{2.5.28}$$

and we claim that

$$\sup_{(x,z)\in k\mathfrak{H}(y,r)} \tilde{u}(x,z) < \infty, \quad k>0. \tag{2.5.29}$$

To prove (2.5.29), recall that the set $\mathcal{R}$ is open, the set $\mathfrak{H}(y,r)$ is bounded and (2.5.29) follows from the concavity of $\tilde{u}$ and $\tilde{u}(x,z) < \infty$ for all $(x,z)\in\mathcal{H}$.

Now, by (2.5.26), (2.5.29) and the definition of domain $\mathcal{H}$, we have further equalities:

$$\begin{aligned}
&\sup_{k>0}\sup_{\tilde{c}\in\mathfrak{A}'(k)}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big]\\
=&\sup_{k>0}\Big\{\sup_{\tilde{c}\in\mathfrak{A}'(k)} \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big]-k\Big\}\\
=&\sup_{k>0}\Big\{\sup_{\tilde{c}\in\widetilde{\mathfrak{A}}(k)} \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big]-k\Big\}\\
=&\sup_{k>0}\Big\{\sup_{(x,z)\in k\mathfrak{H}(y,r)} \tilde{u}(x,z)-k\Big\}\\
=&\sup_{(x,z)\in\mathcal{H}}\{\tilde{u}(x,z)-xy+zr\}.
\end{aligned}$$

On the other hand,

$$\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\sup_{\tilde{c}\in\mathcal{S}_n}\mathbb{E}\Big[\int_0^T \Big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\Big)dt\Big] = \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\Big[\int_0^T V^n(t,\Gamma_t,\omega)dt\Big] \triangleq \tilde{v}^n(y,r),$$

where we define $V^n(t,y,\omega)$ according to the definition of set $\mathcal{S}_n$ as

$$V^n(t,y,\omega) = \sup_{0<x\le n\tilde{w}}\Big[U(t,x)-xy\Big].$$

Consequently, it is sufficient to show that

$$\lim_{n\to\infty} \tilde{v}^n(y,r) = \lim_{n\to\infty} \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\Big[\int_0^T V^n(t,\Gamma_t,\omega)dt\Big] = \tilde{v}(y,r), \quad (y,r)\in\mathcal{R}.$$

Evidently, $\tilde{v}^n(y,r) \le \tilde{v}(y,r)$, for $n \ge 1$. Let $(\Gamma^n)_{n\ge 1}$ be a sequence in $\widetilde{\mathcal{Y}}(y,r)$ such that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_0^T V^n(t,\Gamma^n_t,\omega)dt\Big] = \lim_{n\to\infty} \tilde{v}^n(y,r).$$

Then we can find a sequence $h^n \in conv(\Gamma^n, \Gamma^{n+1}, \ldots)$, $n \ge 1$, converging almost surely to a variable Γ. We have $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$, because the set $\widetilde{\mathcal{Y}}(y,r)$ is closed under convergence in probability.

Now, we claim the sequence of processes $(V^n(\cdot, h^n_\cdot, \omega)^-), n \ge 1$ is uniformly integrable, and in fact, we can rewrite

$$\Big(V^n(t,h^n_t,\omega)\Big)^- = \Big(V^n(t,h^n_t,\omega)\Big)^- 1_{\{h^n_t \le U'(t,\tilde{w}_t)\}} + \Big(V^n(t,h^n_t,\omega)\Big)^- 1_{\{h^n_t > U'(t,\tilde{w}_t)\}}, \tag{2.5.30}$$

and since $V^n(t,y,\omega) = V(t,y)$ for $y \ge U'(t,\tilde{w}_t) \ge U'(t,n\tilde{w}_t)$ by the definition. The argument from Lemma 2.5.9 asserts the uniform integrability of the sequence of processes $\Big(V^n(\cdot,h^n_\cdot,\omega)\Big)^- 1_{\{h^n_\cdot > U'(\cdot,\tilde{w}_\cdot)\}}, n \ge 1$.

On the other hand, by the monotonicity of $(V^n)^-$, we have for all $n > 1$,

$$\Big(V^n(t,h^n_t,\omega)\Big)^- 1_{\{h^n_t \le U'(t,\tilde{w}_t)\}} \le \Big(V^1(t,h^n_t,\omega)\Big)^- 1_{\{h^n_t \le U'(t,\tilde{w}_t)\}} \le \Big(V(t,\,U'(t,\tilde{w}_t))\Big)^- \tag{2.5.31}$$

and by Lemma 2.5.12 the right hand side is integrable in the product space, and hence we conclude the sequence $\Big(V^n(\cdot,h^n_\cdot,\omega)\Big)^- 1_{\{h^n_\cdot \le U'(\cdot,\tilde{w}_\cdot)\}}, n \ge 1$ is also uniformly integrable, and hence our claim holds true. Moreover, we will have the following inequalities:

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\Big[\int_0^T V^n(t,\Gamma_t^n,\omega)dt\Big] \geq & \liminf_{n\to\infty}\mathbb{E}\Big[\int_0^T V^n(t,h_t^n,\omega)dt\Big] \\ \geq & \mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big] \geq \tilde{v}(y,r),\end{aligned}$$

which proves:

$$\tilde{v}(y,r) = \sup_{(x,z)\in\mathcal{H}}\{\tilde{u}(x,z) - xy + zr\}. \tag{2.5.32}$$

For the other equality (2.5.21), define the function $f(x,z)$ from $\mathbb{R}^2$ to $\bar{\mathbb{R}}$ as

$$f(x,z) \triangleq \begin{cases} cl(-\tilde{u}(x,z)) & (x,z)\in cl\mathcal{H}, \\ \infty, & \text{otherwise.}\end{cases} \tag{2.5.33}$$

where $cl(-\tilde{u}(x,z))$ is the lower semicontinuous hull of function $-u(x,z)$. Then f is a proper, convex and lower-semicontinuous function on $\mathbb{R}$ and notice $int(dom(f)) = \mathcal{H}$. By Corollary 12.2.2 in *Rockafella* [66], its Fenchel-Legendre transform is defined by

$$\tilde{f}(y,r) = \sup_{(x,z)\in\mathbb{R}^2}(-xy+zr-f(x,z)) = \sup_{(x,z)\in\mathcal{H}}(-xy+zr+\tilde{u}(x,z)), \quad (y,r)\in\mathbb{R}^2.$$

Observe that if $(y,r)\in\mathcal{R}$, we have $\tilde{f}(y,r) = \tilde{v}(y,r)$ by (2.5.32), and if $(y,r)\notin cl\mathcal{R}$, we have by the increasing property of $\tilde{u}(x,z)$ that

$$\tilde{f}(y,r) \geq s(-x_0y + z_0r) + \tilde{u}(x_0,z_0)$$

for any $s>1$ and fixed $(x_0,z_0)\in\mathcal{H}$. We can therefore conclude that $\tilde{f}(y,r) = \infty$ for $(y,r)\notin cl\mathcal{R}$ since $-x_0y+z_0r>0$ by the definition of $\mathcal{R}$. We can thus apply Theorem 12.2 in *Rockafella* [66] to derive that

$$f(x,z) = \sup_{(y,r)\in\mathbb{R}^2} (-xy + zr - \tilde{f}(y,r)), \quad \forall (x,z) \in \mathbb{R}^2.$$

Again, by Corollary 12.2.2 in *Rockafella* [66] and the fact that $int(dom(\tilde{f})) = int(dom(\tilde{v})) \subseteq \mathcal{R}$, we further have

$$f(x,z) = \sup_{(y,r)\in\mathcal{R}} (-xy + zr - \tilde{v}(y,r)) = -\inf_{(y,r)\in\mathcal{R}} (\tilde{v}(y,r) + xy - zr), \quad \forall (x,z) \in \mathbb{R}^2.$$

In particular, we deduce that relation

$$\tilde{u}(x,z) = \inf_{(y,r)\in\mathcal{R}} \{\tilde{v}(y,r) + xy - zr\}, \quad \forall (x,z) \in \mathcal{H},.$$

$\square$

PROOF OF THEOREM 2.4.2.

It is now sufficient to show the conjugate value function $\tilde{v}$ is $(-\infty,\infty]$-valued on $\mathcal{R}$.

Now, according to the definition of Legendre transform, we have

$$U(t,x) \leq V(t,y) + xy$$

by integration, it is easy to see for any $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$ and $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$, we have

$$\mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big] \leq \mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big] + \mathbb{E}\Big[\int_0^T \tilde{c}_t\Gamma_t dt\Big],$$

from which Proposition 2.5.1 deduces that

$$\tilde{u}(x,z) \leq \tilde{v}(y,r) + xy - zr,$$

and hence we obtain for all $(y, r) \in \mathcal{R}$, we have $\tilde{v}(y, r) > -\infty$ by Lemma 2.5.5.

On the other hand, thanks to conjugate duality (2.5.21) and Bipolar relationship (2.5.9), follow the proofs in Lemma 2.5.7 and Lemma 2.5.14, we also have for each fixed $(y, r) \in \mathcal{R}$

$$\sup_{(x,z)\in k\mathfrak{H}(y,r)} \tilde{u}(x, z) = \inf_{s>0}\{\tilde{v}(sy, sr) + ks\}.$$

The finiteness result (2.5.29) for all $k > 0$ in the proof of Lemma 2.5.14 guarantees the existence of a constant $s(y, r) > 0$, such that $\tilde{v}(sy, sr) < \infty$. $\square$

2.5.2 The Proof of Theorem 2.4.3

Let's move on to the proof of Theorem 2.4.3, to this end, we will need some further lemmas and priori results.

Lemma 2.5.15. *Under assumptions of Theorem* 2.4.3*, we have* $\tilde{v}(y, r)$ *is* $(-\infty, \infty)$*-valued on* $\mathcal{R}$.

Proof. Similar to the proof of Lemma 2.5.5, under the additional Assumption (2.2.9), we can show $\tilde{v}(y, r) < \infty$ if $\tilde{v}(sy, sr) < \infty$ for a constant $s = s(y, r) > 0$. And we have shown that Theorem 2.4.2 asserts the existence of $s = s(y, r) > 0$. $\square$

We wish to draw the readers attention that we can not simply mimic the proofs of Lemma 2.5.8, 2.5.9 and 2.5.10 to obtain the existence and uniqueness of our auxiliary primal Utility Maximization problem (2.3.22). In fact, our successful arguments for the dual problem are hinged on the existence of a bounded process $\tilde{w} \in \widetilde{\mathfrak{A}}(\frac{\bar{p}}{\lambda})$, which is missing in the dual space. In a nutshell, it is more delicate to work out the proof for the existence of the primal optimizer, nevertheless, the prescribed assumptions on the Reasonable Asymptotic Elasticity permits us to interplay the primal optimizer to the optimal solution to some dual problems. To this end, we resort to a further auxiliary optimization problem of the auxiliary dual Utility Minimization problem (2.4.5), and make advantage of the Bipolar results built in Lemma 2.5.7.

Lemma 2.5.16. *Define the auxiliary optimization problem to the auxiliary dual Utility Minimization problem* (2.4.5) *as:*

$$\hat{v}(k) = \inf_{\Gamma \in \widetilde{\mathfrak{Y}}(k)} \mathbb{E}\Big[\int_0^T V(t, \Gamma_t) dt\Big], \tag{2.5.29}$$

where $\widetilde{\mathfrak{Y}}(k)$ is defined in Lemma 2.5.7 *as the bipolar set of $\widetilde{\mathcal{A}}(x,z)$ on the product space for any $(x,z) \in \mathcal{H}$.*

Then, for all $k > 0$, under hypothesis of Theorem 2.4.3*, the value function $\hat{v}(k) < \infty$ for all $k > 0$, and the optimal solution $\hat{\Gamma}(k)$ exists and is unique and $\hat{\Gamma}_t(k) > 0$ for all $t \in [0,T]$. Moreover, for each $k > 0$, and any $\Gamma \in \widetilde{\mathfrak{Y}}(k)$, we have*

$$\mathbb{E}\Big[\int_0^T (\Gamma_t - \hat{\Gamma}_t(k)) I(t, \hat{\Gamma}_t(k)) dt\Big] \leq 0.$$

Proof. According to the definition in Lemma 2.5.7, it is easy to see

$$\begin{aligned}\hat{v}(k) = \inf_{\Gamma \in \widetilde{\mathfrak{Y}}(k)} \mathbb{E}\Big[\int_0^T V(t, \Gamma_t) dt\Big] &\leq \inf_{\Gamma \in \mathfrak{Y}(k)} \mathbb{E}\Big[\int_0^T V(t, \Gamma_t) dt\Big] \\ &= \inf_{(y,r) \in k\mathfrak{R}(x,z)} \tilde{v}(y,r) < \infty, k > 0.\end{aligned}$$

by Lemma 2.5.15.

Taking into account the Bipolar relationship (2.5.11), we have $\widetilde{\mathfrak{Y}}(k)$ is convexly compact in $\mathbb{L}^0_+$, the existence and uniqueness of optimal solution $\hat{\Gamma}(k)$ will follow the similar proof of Theorem 2.4.2.

Now, for $k > 0$, $\epsilon \in (0,1)$ and define $\Gamma^\epsilon_t = (1-\epsilon)\hat{\Gamma}_t(k) + \epsilon\Gamma_t$, for all $t \in [0,T]$, the optimality of $\hat{\Gamma}(k)$ implies

$$\begin{aligned} 0 \leq & \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^T \Big(V(t,\Gamma^\epsilon_t) - V(t,\hat{\Gamma}_t(k))\Big)dt\Big] \\ \leq & \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^T \Big(\hat{\Gamma}_t(k) - \Gamma^\epsilon_t\Big)I(t,\Gamma^\epsilon_t)dt\Big] \\ = & \mathbb{E}\Big[\int_0^T \Big(\hat{\Gamma}_t(k) - \Gamma_t\Big)I(t,\Gamma^\epsilon_t)dt\Big]. \end{aligned} \tag{2.5.30}$$

We claim the family $\Big\{\Big((\Gamma_t - \hat{\Gamma}_t(k))I(t,\Gamma^\epsilon_t)\Big)^-, \epsilon \in (0,1)\Big\}$ is uniformly integrable with respect to $\bar{\mathbb{P}}$, since first

$$\Big((\Gamma_t - \hat{\Gamma}_t(k))I(t,\Gamma^\epsilon_t)\Big)^- \leq \hat{\Gamma}_t(k)I(t,\Gamma^\epsilon_t) \leq \hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k)),\ \forall t \in [0,T].$$

We fix $\epsilon_0 < 1$ and observe that for $\epsilon < \epsilon_0$, we have for each $t \in [0,T]$,

$$\begin{aligned} \Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big| \leq & \Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{\hat{\Gamma}_t(k)\leq y_1\}} \\ + & \Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{\hat{\Gamma}_t(k)\geq \frac{y_2}{1-\epsilon_0}\}} + \Big|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\Big|1_{\{y_1<\hat{\Gamma}_t(k)<\frac{y_2}{1-\epsilon_0}\}}. \end{aligned}$$

Now fix $\epsilon_0 < 1$ and observe that for $\epsilon < \epsilon_0$, recall by Lemma 2.2.2 and Corollary 2.2.3, assumptions on Reasonable Asymptotic Elasticity $AE_0[U] < \infty$ and $AE_\infty[U] < 1$ imply for fixed $\mu > 0$, the existence of constants $C_1 > 0$, $C_2 > 0$, $y_1 > 0$ and $y_2 > 0$ such that

$$\begin{aligned} -V'(t,\mu y) < C_1\frac{V(t,y)}{y} \quad & \text{for } 0 < y \leq y_1, \\ -V'(t,y) < C_2\frac{-V(t,y)}{y} \quad & \text{for } y_2 \leq y. \end{aligned} \tag{2.5.31}$$

Hence, the first term is dominated by

$$\left|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\right|1_{\{\hat{\Gamma}_t(k)\leq y_1\}} \leq \frac{1}{1-\epsilon_0}C_1\,V(t,\hat{\Gamma}_t(k)),$$

and the send term is dominated by

$$\begin{aligned}\left|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\right|1_{\{\hat{\Gamma}_t(k)\geq\frac{y_2}{1-\epsilon_0}\}} \leq & \frac{-1}{1-\epsilon_0}C_2\,V(t,(1-\epsilon)\hat{\Gamma}_t(k)) \\ \leq & \frac{-1}{1-\epsilon_0}C_2\,V(t,\hat{\Gamma}_t(k)).\end{aligned}$$

These two terms are both in $\mathbb{L}^1$ by the finiteness of $\hat{v}(k)$. On the other hand, the third remaining term $\left|\hat{\Gamma}_t(k)I(t,(1-\epsilon)\hat{\Gamma}_t(k))\right|1_{\{y_1<\hat{\Gamma}_t(k)<\frac{y_2}{1-\epsilon_0}\}}$ is dominated by $k\hat{\Gamma}_t(k)1_{\{y_1<\hat{\Gamma}_t(k)<\frac{y_2}{1-\epsilon_0}\}}$ for a constant $k>0$, and it is obviously integrable as well.

Now we can let $\epsilon\to 0$ and apply Dominated convergence theorem and Fatou's lemma to obtain the stated inequality.

To show the optimal solution $\hat{\Gamma}_t(k)>0$ for all $t\in[0,T]$, choose an element $\Gamma_t>0\in\widetilde{\mathfrak{Y}}(k)$ for all $t\in[0,T]$, it is enough to rewrite the inequality (2.5.30) as

$$0\geq\mathbb{E}\Big[\int_0^T\Big(\Gamma_t-\hat{\Gamma}_t(k)\Big)I(t,\Gamma_t^{\epsilon})\mathbf{1}_{\{\hat{\Gamma}_t(k)>0\}}dt\Big]+\mathbb{E}\Big[\int_0^T\Big(\Gamma_t-\hat{\Gamma}_t(k)\Big)I(t,\Gamma_t^{\epsilon})\mathbf{1}_{\{\hat{\Gamma}_t(k)=0\}}dt\Big]. \tag{2.5.32}$$

Now suppose $\bar{\mathbb{P}}\{\hat{\Gamma}_t(k)=0\}>0$, then by the uniform integrability of $\Big\{\Big((\Gamma_t-\hat{\Gamma}_t(k))I(t,\Gamma_t^{\epsilon})\Big)^{-},\epsilon\in(0,1)\Big\}$, let ϵ converges to 0, the second term of (2.5.32) goes to ∞, since $I(t,0)=\infty$ and $\Gamma_t>0$ for all $t\in[0,T]$, and we obtain the contradiction. Hence the conclusion holds. □

Lemma 2.5.17. *Under Assumptions of Theorem* 2.4.3*, the auxiliary dual value function* $\hat{v}(k)$ *is continuously differentiable on* $(0,\infty)$*, and*

$$-k\hat{v}'(k) = \mathbb{E}\Big[\int_0^T \hat{\Gamma}_t(k) I(t, \hat{\Gamma}_t(k)) dt\Big]. \tag{2.5.33}$$

Proof. In order to show $\hat{v}(k)$ is continuously differentiable, notice the convexity property, it is enough to justify that its derivative exists on $(0,\infty)$. Now fix $k>0$, and define the function

$$h(s) \triangleq \mathbb{E}\Big[\int_0^T V(t, \frac{s}{k}\hat{\Gamma}_t(k)) dt\Big].$$

This function is convex and by optimality of $\hat{\Gamma}(k)$ of problem (2.5.29), we have $h(s) \geq \hat{v}(s)$ for all $s>0$ and $h(k) = \hat{v}(k)$. Again, by convexity, we obtain

$$\Delta^- h(k) \leq \Delta^- \hat{v}(k) \leq \Delta^+ \hat{v}(k) \leq \Delta^+ h(k),$$

where Δ^+ and Δ^- denote right- and left-derivatives, respectively. Now

$$\begin{aligned}
\Delta^+ h(k) =& \lim_{\epsilon\to 0} \frac{h(k+\epsilon) - h(k)}{\epsilon} \\
=& \lim_{\epsilon\to 0} \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^T \Big(V(t, \frac{k+\epsilon}{k}\hat{\Gamma}_t(k)) - V(t, \hat{\Gamma}_t(k))\Big) dt\Big] \\
\leq& \liminf_{\epsilon\to 0}\Big(-\frac{1}{k\epsilon}\Big)\mathbb{E}\Big[\int_0^T \epsilon\hat{\Gamma}_t(k) I(t, \frac{k+\epsilon}{k}\hat{\Gamma}_t(k)) dt\Big] \\
=& -\frac{1}{k}\mathbb{E}\Big[\int_0^T \hat{\Gamma}_t(k) I(t, \hat{\Gamma}_t(k)) dt\Big],
\end{aligned}$$

by the Monotone Convergence Theorem.

Similarly, we get

$$\Delta^- h(k) \geq \limsup_{\epsilon\to 0}\mathbb{E}\Big[-\int_0^T \hat{\Gamma}_t(k) I(t, \frac{k-\epsilon}{k}\hat{\Gamma}_t(k)) dt\Big].$$

We can follow the same reasoning as in Lemma 2.5.16 to show the family $\{\Big(\hat{\Gamma}_t(k)I(t,\frac{k-\epsilon}{k}\hat{\Gamma}_t(k))\Big)^-, \epsilon \in (0, 1\wedge k)\}$ is uniformly integrable, and Dominated Convergence Theorem and Fatou's Lemma produce that

$$\Delta^- h(k) \geq -\frac{1}{k}\mathbb{E}\Big[\int_0^T \hat{\Gamma}_t(k)I(t,\hat{\Gamma}_t(k))dt\Big],$$

which completes the proof. $\square$

Lemma 2.5.18. *The auxiliary dual value function $\hat{v}(\cdot)$ has the asymptotic property:*

$$-\hat{v}'(0) = \infty, \qquad -\hat{v}'(\infty) = 0. \tag{2.5.34}$$

Proof. We first show $-\hat{v}'(0) = \infty$, and to this end, we can first derive the result that

$$\hat{v}(0+) \geq \int_0^T V(t, 0+)dt. \tag{2.5.35}$$

To prove the validity of (2.5.35), we observe that for any $k > 0$, by the definition we have

$$\hat{v}(k) = \mathbb{E}\Big[\int_0^T V(t,\hat{\Gamma}_t(k))dt\Big] = \mathbb{E}\Big[\int_0^T V^+(t,\hat{\Gamma}_t(k))dt\Big] - \mathbb{E}\Big[\int_0^T V^-(t,\hat{\Gamma}_t(k))dt\Big],$$

hence, by Fatou's Lemma, firstly, we have

$$\lim_{k\to 0}\mathbb{E}\Big[\int_0^T V^+(t,\hat{\Gamma}_t(k))dt\Big] \geq \mathbb{E}\Big[\int_0^T V^+(t,0+)dt\Big]. \tag{2.5.36}$$

On the other hand, similar to the proof of Lemma 2.5.8, we can show that

$$\mathbb{E}\Big[\int_0^T V^-(t,\hat{\Gamma}_t(1))dt\Big] < \infty,$$

and therefore, by the Monotonicity of function $V^{-}(t,\cdot)$ and Dominated Convergence Theorem, we can easily derive that

$$\lim_{k\to 0}\mathbb{E}\Big[\int_0^T V^{-}(t,\hat{\Gamma}_t(k))dt\Big]=\mathbb{E}\Big[\int_0^T V^{-}(t,0+)dt\Big],$$

which together with (2.5.36) implies that (2.5.35) holds true.

Therefore, if $\int_0^T V(t,0+)dt=\infty$, then we have $\hat{v}(0+)=\infty$, and by convexity, we have $\hat{v}'(0+)=-\infty$.

In the case $\int_0^T V(t,0+)dt<\infty$, we then have

$$-\hat{v}(0+)\geq\lim_{k\to 0}\frac{\hat{v}(0)-\hat{v}(k)}{k}\geq\lim_{k\to 0}\frac{\int_0^T V(t,0+)dt-\mathbb{E}\Big[\int_0^T V(t,\hat{\Gamma}_t(k))dt\Big]}{k},$$

and hence we have

$$\begin{aligned}-\hat{v}(0+)&\geq\lim_{k\to 0}\frac{\mathbb{E}\Big[\int_0^T V(t,0+)dt\Big]-\mathbb{E}\Big[\int_0^T V(t,\hat{\Gamma}_t(k))dt\Big]}{k}\\&\geq\lim_{k\to 0}\mathbb{E}\Big[\int_0^T\hat{\Gamma}_t(1)I(t,k\hat{\Gamma}_t(1))dt\Big]=\infty,\end{aligned}$$

by the Monotone Convergence Theorem.

We can now turn to show that $-\hat{v}'(\infty)=0$, and since the function $-\hat{v}$ is concave and increasing, there is a finite positive limit

$$-\hat{v}'(\infty)\triangleq\lim_{k\to\infty}-\hat{v}'(y).$$

By the definition of Legendre Transform, we clearly have for any $y>0$,

$$-V(t,y)\leq -U(t,x)+xy,\text{for all } x>0,$$

and then for any $\epsilon > 0$, we always have:

$$\begin{aligned}0 \le -\hat{v}'(\infty) = \lim_{k\to\infty}\frac{-\hat{v}(k)}{k} &= \lim_{k\to\infty}\frac{\mathbb{E}\Big[\int_0^T -V(t,\hat{\Gamma}_t(k))dt\Big]}{k}\\ &\le \lim_{k\to\infty}\frac{\mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)dt\Big]}{k} + \lim_{k\to\infty}\frac{\big\langle \epsilon\tilde{w},\hat{\Gamma}(k)\big\rangle}{k}.\end{aligned}$$

Now, recall that for each fixed $(x,z) \in \mathcal{H}$, there exists a constant $\lambda(x,z) > 0$ such that we have $\tilde{w}_t \in \widetilde{\mathcal{A}}(\frac{\bar{p}}{\lambda}x, \frac{\bar{p}}{\lambda}z)$, and by the definition of $\widetilde{\mathfrak{Y}}(k)$, we can see the second term above has

$$\lim_{k\to\infty}\frac{\big\langle \epsilon\tilde{w},\hat{\Gamma}(k)\big\rangle}{k} \le \lim_{k\to\infty}\frac{\epsilon\frac{\bar{p}}{\lambda}k}{k} = \epsilon\frac{\bar{p}}{\lambda}.$$

As for the first term, we claim that $\mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)dt\Big] < \infty$ for each fixed ϵ small enough, without loss of generality, we just need to consider that $\epsilon < \bar{x}$, and then we will apply Corollary 2.2.3 again, and since there exists a constant x_0 such that for all $t \in [0,T]$,

$$U(t,\frac{\epsilon}{\bar{x}}x) > (\frac{\epsilon}{\bar{x}})^{-\frac{\gamma}{1-\gamma}}U(t,x) \;\text{ for } 0 < x \le x_0,$$

we will have

$$\begin{aligned}\mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)dt\Big] &= \mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t > x_0\}}dt\Big]\\ &\quad + \mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t \le x_0\}}dt\Big]\\ &\le \mathbb{E}\Big[\int_0^T -U(t,\epsilon\tilde{w}_t)1_{\{\bar{x}\tilde{w}_t > x_0\}}dt\Big]\\ &\quad + (\frac{\epsilon}{\bar{x}})^{-\frac{\gamma}{1-\gamma}}\mathbb{E}\Big[\int_0^T -U(t,\bar{x}\tilde{w}_t)dt\Big] < \infty\end{aligned}$$

by the fact that $\tilde{w} \preceq 1$ and the Assumption (2.3.4).

Hence, we conclude that

$$0 \leq -\hat{v}'(\infty) = \lim_{k\to\infty} \frac{-\hat{v}(k)}{k} \leq \epsilon\frac{\bar{p}}{\lambda},$$

and consequently, we have $-\hat{v}'(\infty) = 0$ by letting ϵ goes to 0. $\square$

Lemma 2.5.19. *Under assumptions of Theorem 2.4.3, for any $(x, z) \in \mathcal{H}$, suppose k satisfies $1 = -\hat{v}'(k)$ where $\hat{v}(k)$ is the value function of the auxiliary dual optimization problem (2.5.29), then $\tilde{c}^*_t(x, z) \triangleq I(t, \hat{\Gamma}_t(k))$ is the unique (in the sense of $\equiv$ in $\mathbb{L}^0_+$) optimal solution to problem (2.3.22), moreover we have $\tilde{c}^*_t(x, z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$.*

Proof. Lemma 2.5.17 asserts

$$\left\langle \tilde{c}^*(x, z), \hat{\Gamma}(k) \right\rangle = -k\hat{v}'(k) = k.$$

And for any $\Gamma \in \widetilde{\mathfrak{Y}}(k)$, by Lemma 2.5.16, we have

$$\left\langle \tilde{c}^*(x, z), \Gamma(k) \right\rangle \leq \left\langle \tilde{c}^*(x, z), \hat{\Gamma}(k) \right\rangle = k.$$

Hence, we get first $\tilde{c}^*_t(x, z) \in \widetilde{\mathcal{A}}(x, z)$ by the Bipolar relationship (2.5.11).

Now, for any $\tilde{c} \in \widetilde{\mathcal{A}}(x, z)$, we have

$$\begin{aligned} \left\langle \tilde{c}, \hat{\Gamma}(k) \right\rangle \leq& k, \\ U(t, \tilde{c}_t) \leq& V(t, \hat{\Gamma}_t(k)) + \tilde{c}_t\hat{\Gamma}_t(k), \ \forall t \in [0, T]. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}\Big[\int_0^T U(t, \tilde{c}_t)dt\Big] \leq& \hat{v}(k) + k = \mathbb{E}\Big[\int_0^T \Big(V(t, \hat{\Gamma}_t(k)) + \hat{\Gamma}_t I(t, \hat{\Gamma}_t(k))\Big)dt\Big] \\ =& \mathbb{E}\Big[\int_0^T U(t, I(\hat{\Gamma}_t(k)))dt\Big] = \mathbb{E}\Big[\int_0^T U(t, \tilde{c}^*_t)dt\Big], \end{aligned} \tag{2.5.37}$$

which shows the optimality of $\tilde{c}^*$. The uniqueness of the optimal solution follows from the strict concavity of the function U.

Moreover, under assumptions of Theorem 2.4.3, for any pair $(x, z) \in \mathcal{H}$, by the fact that $\widetilde{\mathfrak{Y}}(k)$ is convexly compact and $\hat{\Gamma}_t(k)$ is bounded in probability, we actually have the optimal solution $\tilde{c}^*_t(x, z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$ since $\hat{\Gamma}_t(k)$ is bounded in probability if and only if $\hat{\Gamma}_t(k)$ is finite $\bar{\mathbb{P}}$-a.s. and by definition, we know $I(t, x) > 0$ for $x < \infty$. $\square$

For the proof of Theorem 2.4.3, we shall also need the following lemma.

Lemma 2.5.20. *Assume that the assumptions of Proposition of* 2.5.1 *hold true. Let the sequences* $(y^n, r^n) \in \mathcal{R}$ *and* $\Gamma^n \in \widetilde{\mathcal{Y}}(y^n, r^n)$, $n \geq 1$, *converges to* $(y, r) \in \mathbb{R}^2$ *and* $\Gamma \in \mathbb{L}^0_+$, *respectively. If* Γ *is a strictly positive random variable, then* $(y, r) \in \mathcal{R}$ *and* $\Gamma \in \widetilde{\mathcal{Y}}(y, r)$.

Proof. Let $(x, z) \in cl\mathcal{H}$ and $\tilde{c} \in \widetilde{\mathcal{A}}(x, z)$ Let $(x, z) \in cl\mathcal{H}$, the proof of Lemma 2.5.2 states there exists $\tilde{c} \in \widetilde{\mathcal{A}}(x, z)$ such that

$$\bar{\mathbb{P}}[\tilde{c} \succ 0] > 0.$$

According to Proposition 2.5.1, we can get

$$0 < \left\langle \tilde{c}, \Gamma \right\rangle \leq xy - zr,$$

by Fatou's lemma.

As (x, z) is an arbitrary element in $cl\mathcal{H}$, it implies that $(y, r) \in \mathcal{R}$. The conclusion that $\Gamma \in \widetilde{\mathcal{Y}}(y, r)$ holds by applying Fatou's lemma and Proposition 2.5.1. $\square$

PROOF OF THEOREM 2.4.3.

We first show the dual value function $\tilde{v}(y,z)$ is continuously differentiable on $\mathcal{R}$. Theorem 4.1.1 and 4.1.2 in *Hiriart-Urruty and Lemaréchal* [34] gives the equivalence between the above statement and the fact that the value function $\tilde{u}(x,z)$ is strictly concave on $\mathcal{H}$. Since U is a strictly concave function, to show the value function is strictly concave is equivalent to show for any two distinct points $(x_i, z_i) \in \mathcal{H}$, $i = 1, 2$, the optimal consumption policies are different:

$$\bar{\mathbb{P}}[\tilde{c}^*(x_1, z_1) \neq \tilde{c}^*(x_2, z_2)] > 0,$$

which is equivalent to Assumption (2.4.2).

As for the remaining piece of the proof, it amounts to show the assertion (ii) hold, and recall $\hat{\Gamma}(k)$ is the optimal solution of the auxiliary dual problem (2.5.29), such that

$$\hat{\Gamma}_t(k) = U'(t, \tilde{c}_t^*(x,z)), \quad \forall t \in [0,T], \quad k = \left\langle \tilde{c}^*(x,z), \hat{\Gamma}(k) \right\rangle.$$

By the definition that $\widetilde{\mathfrak{Y}}(k)$ is closed with respect to convergence in measure $\bar{\mathbb{P}}$, there exists a sequence $(y^n, r^n) \in k\mathfrak{R}(x,z)$ such that $\Gamma^n \in \widetilde{\mathcal{Y}}(y^n, r^n)$ and Γ^n converges to $\hat{\Gamma}(k)$ $\bar{\mathcal{P}}$-a.s. by passing to a subsequence if necessary, and since set $k\mathfrak{R}(x,z)$ is bounded, there exists a further subsequence (y^n, r^n) converges to $(y, r) \in \mathbb{R}^2$. By passing to this further subsequence, as we have shown $\bar{\mathbb{P}}[\hat{\Gamma}(k) \succ 0] = 1$, we will have $(y, r) \in k\mathfrak{R}(x,z)$ such that $\hat{\Gamma}(k) \in \widetilde{\mathcal{Y}}(y, r)$ due to Lemma 2.5.20. Moreover, for this pair $(y, r) \in \mathcal{R}$, by Fatou's Lemma and Proposition 2.5.1, we have the equality that

$$xy - zr = k = \left\langle \tilde{c}^*(x,z), \hat{\Gamma}(k) \right\rangle. \tag{2.5.38}$$

And we have the corresponding optimizer $\Gamma_t^*(y,r)$ of (2.4.5) verifies

$$\Gamma_t^*(y,r) = \hat{\Gamma}_t(k) = U'(t, \tilde{c}^*(x,z)), \tag{2.5.39}$$

because on one hand, we have $\hat{\Gamma}(k) \in \widetilde{\mathcal{Y}}(y,r)$, hence

$$\mathbb{E}\int_0^T V(t, \Gamma_t^*(y,r)) = \inf_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\int_0^T V(t, \Gamma_t(y,r)) \le \mathbb{E}\int_0^T V(t, \hat{\Gamma}_t(k)),$$

and on the other hand, we have

$$\begin{aligned}\mathbb{E}\int_0^T V(t, \hat{\Gamma}_t(y,r)) &= \inf_{\Gamma \in \widetilde{\mathfrak{Y}}(k)} \mathbb{E}\int_0^T V(t, \Gamma_t(y,r)) \\ &\le \inf_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\int_0^T V(t, \Gamma_t(y,r)) = \mathbb{E}\int_0^T V(t, \Gamma_t^*(y,r)).\end{aligned}$$

By the equality

$$U(t, \tilde{c}_t^*(x,z)) = V(t, \hat{\Gamma}_t(k)) + \tilde{c}_t^*(x,z)\hat{\Gamma}_t(k),$$

we can conclude $(y,r) \in \partial\tilde{u}(x,z)$ by Theorem 23.5 of *Rockafellar* [66], since we have

$$\tilde{u}(x,z) = \tilde{v}(y,z) + xy - zr \tag{2.5.40}$$

In particular, we get

$$\partial\tilde{u}(x,z) \cap \mathcal{R} \neq \varnothing. \tag{2.5.41}$$

Similar to the proof of Theorem 2 in *Hugonnier and Kramkov* [35], we can actually show

$$\partial\tilde{u}(x,z) \subset \mathcal{R}.$$

For any $(y,r) \in \partial\tilde{u}(x,z)$, we can find a sequence $(y^n, r^n) \in \partial\tilde{u}(x,z) \cap \mathcal{R}$ converging to (y,r) by (2.5.41) and the fact that $\partial\tilde{u}(x,z)$ is closed and convex. Since $U'(\cdot, \tilde{c}^*_\cdot(x,z))$ is strictly positive and we know $U'(\cdot, \tilde{c}^*_\cdot(x,z)) \in \widetilde{\mathcal{Y}}(y,r)$. Lemma 2.5.20 now infers $(y,r) \in \mathcal{R}$.

Conversely, for any $(y,r) \in \partial\tilde{u}(x,z)$, then

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \Big| V(t, \Gamma^*_t(y,r)) + \tilde{c}^*_t(x,z)\Gamma^*_t(y,r) - U(t, \tilde{c}^*_t(x,z))\Big| dt\Big] \\
=&\mathbb{E}\Big[\Big(\int_0^T V(t, \Gamma^*_t(y,r)) + \tilde{c}^*_t(x,z)\Gamma^*_t(y,r) - U(t, \tilde{c}^*_t(x,z)) dt\Big)\Big] \\
\leq&\tilde{v}(y,r) + xy - zr - \tilde{u}(x,z) = 0,
\end{aligned}$$

which infers (2.5.38) and (2.5.39). □

2.6 The Special Case When $\mathcal{E}$ is Replicable

2.6.1 The One Dimensional Primal Value Function

We begin this section by describing the special case when the random variable defined by

$$\mathcal{E} \triangleq \int_0^T w_t dt = \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} dt$$

is replicable, i.e., there exists a constant $\bar{r} > 0$ such that

$$\big\langle w, Y \big\rangle = \bar{r}, \quad \forall Y \in \mathcal{M}, \tag{2.6.1}$$

where $\mathcal{M}$ is the set of all equivalent local martingale measure densities. We will make some comments on the suggestion made by some previous papers for the extension of their work to the incomplete markets, and redefine the modified dual problem for the utility maximization problem with habit formation in incomplete markets driven by geometric Brownian motions when the discounting factors are restricted to be deterministic functions.

In the light of the fact that $\bar{p} = \underline{p} = \bar{r}$ where $\bar{p}$ and $\underline{p}$ are defined in (2.3.19) and (2.3.20), we see the closure of domain $\mathcal{R}$ for the pair (y, r) defined in (2.4.3) shrinks into a line, and hence the set $\mathcal{R}$ is not well defined. In order to build the similar conjugate duality, we will need to reconsider our primal Utility Maximization problem and the corresponding dual optimization problem, and embed our problem into the framework of *Kramkov and Schachermayer* [48], [49]. In particular, instead of defining the primal value function on two variables of initial wealth $x > 0$ and initial standard of living $z \geq 0$, we can reduce its dimension and define the new variable $\tilde{x} = z - z\bar{r}$. According to Lemma 2.3.1, the effective domain $\bar{\mathcal{H}}$ for x and z mandating the constraint that $x > z\bar{r}$ can therefore be transformed to the constraint on the choice of $\tilde{x}$ as $\tilde{x} > 0$.

Recall the auxiliary dual set $\widetilde{\mathcal{M}}$ is defined by

$$\widetilde{\mathcal{M}} = \Big\{\Gamma \in \mathbb{L}_+^0 : \Gamma_t = Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} Y_s ds \Big| \mathcal{F}_t\Big],\ \ \forall t \in [0, T],\ \ \forall Y \in \mathcal{M}\Big\}.$$

In order to invoke the one dimensional convex duality analysis in *Kramkov and Schachermayer* [48], [49], we first define the domain of the auxiliary processes $\tilde{c}$ depending on the value of $\tilde{x}$ as

$$\widetilde{\mathcal{A}}(\tilde{x}) \triangleq \Big\{\tilde{c} \in \mathbb{L}_+^0 :\ \langle \tilde{c}, \Gamma \rangle \leq \tilde{x},\quad \forall \Gamma \in \widetilde{\mathcal{M}}\Big\},\quad \text{for } \tilde{x} > 0. \tag{2.6.2}$$

Then, it is clear that we have the equivalence that $\bar{\mathcal{A}}(x,z) = \widetilde{\mathcal{A}}(\tilde{x})$ if we take $\tilde{x} = x - z\bar{r}$.

It is now ready to define the **Auxiliary Primal Utility Maximization Problem** with respect to $\tilde{c}$ as

$$\tilde{u}(\tilde{x}) \triangleq \sup_{\tilde{c}\in\widetilde{\mathcal{A}}(\tilde{x})} \mathbb{E}\Big[\int_0^T U(t,\tilde{c}_t)dt\Big], \quad \tilde{x} > 0. \tag{2.6.3}$$

2.6.2 The Dual Optimization Problem and Main Results

We denote the set $\widetilde{\mathcal{Y}}(1)$ as the bipolar set of $\widetilde{\mathcal{M}}$, i.e.,

$$\widetilde{\mathcal{Y}}(1) = \widetilde{\mathcal{M}}^{\circ\circ}, \tag{2.6.4}$$

then we know the set $\widetilde{\mathcal{Y}}(1)$ is the closure of the solid hull of $\widetilde{\mathcal{M}}$, and we are ready to define the *Auxiliary Dual Utility Optimization Problem* as

$$\tilde{v}(y) = \inf_{\Gamma\in y\widetilde{\mathcal{Y}}(1)} \mathbb{E}\Big[\int_0^T V(t,\Gamma_t)dt\Big]. \tag{2.6.5}$$

Then, we have the following theorems which consist of the main results in this section

Theorem 2.6.1. *Assume conditions* (2.2.5)*,* (2.2.7)*,* (2.3.4) *hold. Assume also that* (2.2.11)*,* (2.2.12) *and* (2.2.10)*,* (*i.e.*, $AE_0[U] < \infty$) *hold true together with*

$$\tilde{u}(\tilde{x}) < \infty \qquad for\ some\ \tilde{x} > 0, \tag{2.6.6}$$

we will have:

(i) The value function $\tilde{u}(\tilde{x})$ *takes value* $(-\infty, \infty)$ *for all* $\tilde{x} > 0$, $\tilde{v}(y)$ *takes value* $(-\infty, \infty]$ *for all* $y > 0$. *And there exists a constant* $y_0 > 0$ *such that* $\tilde{v}(y) < \infty$ *for* $y > y_0$. *The value functions* $\tilde{u}$ *and* $\tilde{v}$ *are conjugate,*

$$\tilde{v}(y) = \sup_{\tilde{x}>0}[\tilde{u}(\tilde{x}) - \tilde{x}y], \quad y > 0,$$
$$\tilde{u}(\tilde{x}) = \sup_{y>0}[\tilde{v}(y) + \tilde{x}y], \quad \tilde{x} > 0.$$

(ii) The function $\tilde{u}$ *is continuously differentiable on* $(0, \infty)$ *and the function* $\tilde{v}$ *is strictly convex on* $\{\tilde{v} < \infty\}$.

(iii) The functions $\tilde{u}'$ *and* $\tilde{v}'$ *satisfy*

$$\tilde{u}'(0) = \lim_{\tilde{x}\to 0} \tilde{u}'(\tilde{x}) = \infty, \quad -\tilde{v}'(\infty) = \lim_{y\to\infty} -\tilde{v}'(y) = 0.$$

(iv) If $\tilde{v}(y) < \infty$, *then the optimal solution* $\Gamma_*(y) \in y\widetilde{\mathcal{Y}}(1)$ *to* (2.6.5) *exists and is unique.*

Theorem 2.6.2. *We now assume in addition to conditions of Theorem* 2.6.1 *that Assumption* (2.2.9) *holds, (i.e.,* $AE_\infty[U] < 1$*). Then in addition to assertions of Theorem* 2.6.1, *we also have:*

(i) $\tilde{v}(y) < \infty$, *for all* $y > 0$. *The value functions* $\tilde{u}$ *is continuously differentiable on* $(0, \infty)$ *and* $\tilde{v}$ *is continuously differentiable on* $(0, \infty)$ *and the functions* $\tilde{u}'$ *and* $-\tilde{v}'$ *are strictly decreasing and satisfy*

$$\tilde{u}'(\infty) = \lim_{\tilde{x}\to\infty} \tilde{u}'(\tilde{x}) = 0, \quad -\tilde{v}'(0) = \lim_{y\to 0} -\tilde{v}'(y) = \infty.$$

(ii) The optimal solution $\tilde{c}^*(\tilde{x})$ *to* (2.6.3) *exists and is unique. If* $\Gamma^*(y)$ *is the optimal solution to* (2.6.5), *where* $y = \tilde{u}'(\tilde{x})$, *we have the dual relation*

$$\tilde{c}_t^*(\tilde{x}) = I(t, \Gamma_t^*(y)), \qquad \Gamma_t^*(y) = U'(t, \tilde{c}_t(\tilde{x})).$$

(iii) For the choice of initial wealth x *and initial standard of living* z *such that* $(x, z) \in \bar{\mathcal{H}}$, *i.e.,* $x > z\bar{r}$, *we have the optimal solution to our primal utility maximization problem* (2.3.10) *exists and is unique, moreover,*

$$c_t^*(x, z) - Z_t^*(x, z) = \tilde{c}_t^*(\tilde{x}), \quad \forall t \in [0, T],$$

where we have $x - z\bar{r} = \tilde{x}$.

The proofs of Theorem 2.6.1 and part (i), (ii) of Theorem (2.6.2) are very similar to the proofs of Theorem (2.4.2) and (2.4.3) in Section 2.5 for the case when $\mathcal{E} = \int_0^T w_t dt = \int_0^T e^{\int_0^t (\delta_v - \alpha_v) dv} dt$ is not replicable, and therefore will be omitted here for the sake of the length of this dissertation. As for part (iii) of Theorem (2.6.2), we just recall that if we have $x - z\bar{r} = \tilde{x} > 0$, then $\bar{\mathcal{A}}(x, z) = \widetilde{\mathcal{A}}(\tilde{x})$ and there is a one-to-one correspondence between set $\mathcal{A}(x, z)$ for the consumption rate process c_t and the set $\bar{\mathcal{A}}(x, z)$ for the auxiliary process $\tilde{c}_t$.

Remark 2.6.1. *When the financial market is assumed to be complete such that* $\mathcal{M} = \{\mathbb{Q}\}$, *the dual problem can be defined on the unique equivalent local martingale measure density process* $Y_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$, *and we indeed have the relation that*

$$c_t^*(x, z) - Z_t^*(x, z) = I(t, yY_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} yY_s ds \Big| \mathcal{F}_t\Big]),$$

for $y = \tilde{u}'(x - z\bar{r})$.

We are now interested in a closer look at the dual domain $\widetilde{\mathcal{Y}}(1) = \widetilde{\mathcal{M}}^{\circ\circ}$. Recall the construction of the auxiliary set $\widetilde{\mathcal{M}}$, we know that each element $\Gamma \in \widetilde{\mathcal{M}}$ is a linear transform of the equivalent local martingale measure density process $Y \in \mathcal{M}$ given by

$$\Gamma_t = Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} Y_s ds \Big| \mathcal{F}_t\Big], \forall t \in [0, T].$$

A natural question is whether the dual domain $\widetilde{\mathcal{Y}}(y)$ can be fully characterized by the same linear transform of supermartingale deflators defined in *Kramkov and Schachermayer* [48], [49]. More precisely, if we define the conventional set $\mathcal{Y}(y)$ as

$$\begin{aligned}\mathcal{Y}(y) \triangleq \Big\{ & Y \in \mathbb{L}^0_+ : Y_0 = y, XY \text{ is a supermartingale for } X = x + H \cdot S \\ & \text{where } H \text{ is admissible}\Big\},\end{aligned}$$

and define the auxiliary set $\bar{\mathcal{Y}}(y)$ by

$$\begin{aligned}\bar{\mathcal{Y}}(y) = solid\Big(\Big\{ & \Gamma \in \mathbb{L}^0_+ :\ \Gamma_t = Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} Y_s ds \Big| \mathcal{F}_t\Big], \forall t \in [0, T] \\ & \text{for } Y \in \mathcal{Y}(y)\Big\}\Big).\end{aligned} \tag{2.6.7}$$

Then

$$\boxed{\textit{Question}: \text{Is it true that } \widetilde{\mathcal{Y}}(1) = \bar{\mathcal{Y}}(1) \text{ ?}} \tag{2.6.8}$$

If the answer is TRUE, then we can always redefine the dual problem over the set of supermartingale deflator Y_t instead of the abstract auxiliary process Γ_t, i.e., we can define the dual optimization problem as:

$$v(y) = \inf_{Y \in \bar{\mathcal{Y}}(y)} \mathbb{E}\Big[\int_0^T V(t, Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} Y_s ds \Big| \mathcal{F}_t\Big])\Big], \tag{2.6.9}$$

and if the dual optimizer $\Gamma_t^*(y)$ in (2.6.5) exists, we should always have the dual optimizer $Y_t^*(y)$ to the problem (2.6.9) exists and we have the equivalence that

$$\Gamma_t^*(y) = Y_t^* + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} Y_s^* ds \Big| \mathcal{F}_t\Big].$$

Unfortunately, the answer to the question (2.6.8) is not always yes. It is important to stress out that the set $\bar{\mathcal{Y}}(y)$ is not necessarily closed due to the conditional expectation of the future path integral with respect to Y_t. Actually, the correct statement is that the dual domain $\widetilde{\mathcal{Y}}(y)$ is the closure of set $\bar{\mathcal{Y}}(y)$ under convergence in probability with respect to the finite measure $\bar{\mathbb{P}}$:

Lemma 2.6.3.

$$\widetilde{\mathcal{Y}}(y) = \overline{\bar{\mathcal{Y}}(y)}. \tag{2.6.10}$$

where $\overline{\bar{\mathcal{Y}}(y)}$ *denote the closure of set* $\bar{\mathcal{Y}}(y)$ *on the product space* $\mathbb{L}_+^0$.

Proof. Without loss of generality, we just need to check the case when $y = 1$. First, it is trivial to show that $\widetilde{\mathcal{Y}}(1) \subseteq \overline{\bar{\mathcal{Y}}(1)}$ holds, since we clearly have $\widetilde{\mathcal{M}} \subseteq \overline{\bar{\mathcal{Y}}(1)}$, and the set $\overline{\bar{\mathcal{Y}}(1)}$ is closed, solid and convex set containing $\widetilde{\mathcal{M}}$.

To prove the other direction inclusion, it amounts to verify that $\overline{\bar{\mathcal{Y}}(1)} \subseteq \widetilde{\mathcal{A}}(1)^{\circ}$. By the definition, for each $\Gamma \in \overline{\bar{\mathcal{Y}}(1)}$, we can find a sequence $\Big(\Gamma^n\Big)_{n\geq 1}$

converges to Γ a.s. in $\bar{\mathbb{P}}$ and $\Gamma^n \preceq \Gamma'^n \in \bar{\mathcal{Y}}(1)$ such that there exists $Y'^1 \in \mathcal{Y}(1)$ with $\Gamma'^n_t = Y'^n_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} Y'^n_s ds \Big| \mathcal{F}_t\Big]$, $\forall t \in [0,T]$. So for each $\tilde{c} \in \widetilde{\mathcal{A}}(1)$, there exists a pair $(x,z) \in \bar{H}$ with $x - z\bar{r} = \tilde{x} = 1$, and by Proposition 2.3.3, there exists $c \in \mathcal{A}(x,z)$ and we have

$$\big\langle \tilde{c}, \Gamma \big\rangle = \big\langle \tilde{c}, \lim_n \Gamma^n \big\rangle \le \big\langle \tilde{c}, \lim_n \Gamma'^n \big\rangle = \lim_n \big\langle c, Y'^n \big\rangle - z\bar{r} \le x - z\bar{r} = \tilde{x} = 1,$$

and our claim holds, where we used Fatou's lemma and the fact that $\mathcal{Y}(1) = \mathcal{M}^{\circ\circ}$ and Consumption Budget Constraint to conclude that $\big\langle c, Y'^n \big\rangle \le x - z\bar{r}$ for each $Y^n \in \mathcal{Y}(1)$. Now, the fact that $\widetilde{\mathcal{Y}}(1) = \widetilde{\mathcal{A}}(1)^\circ$ implies that $\overline{\bar{\mathcal{Y}}(1)} \subseteq \widetilde{\mathcal{Y}}(1)$ which completes the proof. $\square$

It is generally very difficult to find the sufficient condition on the financial market such that the associated auxiliary set $\bar{\mathcal{Y}}(y)$ is closed. And this means the optimal dual solution for the problem (2.6.5) is an abstract process in general, which does not provide any explicit financial intuitions with respect to the market. This is the unique and difficult aspect of our original utility maximization problem.

However, it is still possible to reveal the vein of its abstract definition in some special cases and provide some financial explanations to the dual domain. One special case we want to discuss in this section is to assume the discounting processes δ_t and α_t satisfies the condition that $\delta_t - \alpha_t$ is a deterministic function in time t.

In this special case, it is clear that we can rewrite the auxiliary set $\widetilde{\mathcal{M}}$

as

$$\widetilde{\mathcal{M}} = \Big\{\Gamma \in \mathbb{L}^0_+ : \Gamma_t = Y_t\Big(1 + \delta_t \int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} ds\Big),\ \forall Y \in \mathcal{M}\Big\}.$$

And we will define another auxiliary dual domain by

$$\widehat{\mathcal{Y}}(y) = \Big\{\Gamma \in \mathbb{L}^0_+ : \Gamma_t = Y_t\Big(1 + \delta_t \int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} ds\Big),\ \forall Y \in solid(\mathcal{Y}(y))\Big\}. \tag{2.6.11}$$

We want to show the following lemma holds,

Lemma 2.6.4.

$$\widetilde{\mathcal{Y}}(y) = \widehat{\mathcal{Y}}(y). \tag{2.6.12}$$

Proof. Again, it is enough for us to prove the conclusion for $y = 1$. For one direction, since the set $\mathcal{Y}(1)$ is closed, convex and solid, from the definition, it is also true that the set $\widehat{\mathcal{Y}}(1)$ is closed, convex and solid on $\mathbb{L}^0_+$. Notice again that $\widetilde{\mathcal{M}} \subseteq \widehat{\mathcal{Y}}(1)$, we can conclude that $\widetilde{\mathcal{Y}}(1) \subseteq \widehat{\mathcal{Y}}(1)$, as $\widetilde{\mathcal{Y}}(1)$ is the smallest closed, convex and solid set containing $\widetilde{\mathcal{M}}$.

One the other hand, by the fact that $solid(\mathcal{Y}(1)) = \mathcal{M}^{\circ\circ}$, for any $\Gamma_t = Y_t K_t \in \widehat{\mathcal{Y}}(1)$ where we denote $K_t = \Big(1 + \delta_t \int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} ds\Big)$, there exists a sequence of processes $Y^n \in \mathcal{M}$ such that Y^n converges to Y. Therefore, we have for any $\tilde{c} \in \widetilde{\mathcal{A}}(1)$

$$\big\langle \tilde{c}, \Gamma\big\rangle = \big\langle \tilde{c}, YK\big\rangle \leq \lim_n \big\langle \tilde{c}, Y^n K\big\rangle,$$

and since Y^n is a true martingale, we have

$$Y^n_t K_t = Y^n_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} Y^n_s ds \Big| \mathcal{F}_t\Big].$$

Consequently, we can derive that the existence of a pair $(x, z) \in \bar{H}$ such that $x - z\bar{r} = 1$ and $c \in \mathcal{A}(x, z)$, and

$$\langle \tilde{c}, Y^n K \rangle = \langle c, Y^n \rangle - z\bar{r} \leq x - z\bar{r} = 1.$$

We proved that $\widehat{\mathcal{Y}}(1) \subseteq \widetilde{\mathcal{A}}(1)^\circ$, and we complete the proof by the fact that $\widetilde{\mathcal{Y}}(y) = \widetilde{\mathcal{A}}(1)^\circ$. $\square$

The statement of Lemma 2.6.4 is easy, but it gives a very nice characterization of our dual domain, and it describes that under the assumption that $\delta_t - \alpha_t$ is deterministic, each element $\Gamma_t(y)$ in the abstract dual domain $\widetilde{\mathcal{Y}}(y)$ is actually the product of the discounted supermartingale $D_t Y_t(y)$ and the unique discounting stochastic process $K_t = \left(1 + \delta_t \int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} ds\right)$, where the optional process D_t takes values in $[0, 1]$ and $Y_t(y) \in \mathcal{Y}(y)$, here the set $\mathcal{Y}(y)$ is conventionally defined as the set of supermartingale deflators with respect to the admissible wealth process as in *Kramkov and Schachermayer* [48], [49]. Furthermore, it enables us to find some examples of the explicit form of the optimal dual process.

2.6.3 An Example in the Itô Process Market Model

In this section, we adopt the same incomplete market model driven by Itô processes in the framework of *Karatzas, Lehoczky, Shreve and Xu* [40], and we still assume $\delta_t - \alpha_t$ is a deterministic function in time t.

Specifically, we consider the financial market with one risk-less bond

S^0 modeled by

$$dS_t^0 = r_t S_t^0 dt, \quad S_0^0 = 1, \tag{2.6.13}$$

and there are m stocks whose price process evolve as

$$dS_t^i = S_t^i b_t^i dt + \sum_{j=1}^{d} S_t^i \sigma_t^{ij} dW_t^j, \quad i = 1, \ldots, m. \tag{2.6.14}$$

Here $W = (W_t)_{0 \le t \le T}$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and we denote $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ the $\mathbb{P}$-augmentation of the filtration generated by W. We assume generally $d \ge m$, i.e., the number of sources of uncertainty in the model is at least as large as the number of stocks available for investment.

We assume the interest rate r_t and the stock drift vector b_t are progressively measurable with respect to $\mathcal{F}_t$ and satisfy $\int_0^T \|b_t\| dt < \infty$ and $\int_0^T |r_t| dt \le L$ a.s. for some constant $L > 0$. The volatility matrix σ_t is also progressively measurable with respect to $\mathcal{F}_t$ and we assume the relative risk process

$$\theta_t \triangleq \sigma_t^\top (\sigma_t \sigma_t^\top)^{-1} [b_t - r_t 1_m] \tag{2.6.15}$$

is well defined. Moreover, we assume $\int_0^T \|\theta_t\|^2 dt < \infty$, a.s. under $\mathbb{P}$.

To be consistent with Theorem 2.6.1 and 2.6.2 and Lemma 2.6.4, we assume our financial market satisfies the $NFLVR$ condition, i.e., $\mathcal{M} \neq \emptyset$.

In incomplete Itô processes markets, *Detemple and Zapatero* [25] and *Egglezos and Karatzas* [26] made the suggestion that it is possible to perform the same program brought up by *Karatzas, Lehoczky, Shreve and Xu* [40], which is to complete the market with some fictitious stocks and invest in a least favorable manner such that the investor does not invest in the additional stocks at all. For the utility maximization problem with consumption habit formation, their dual optimization problem eventually is written in the following form

$$\inf_{y>0,\nu\in H(\sigma)} \mathbb{E}\Big[\int_0^T V\Big(yY_t^\nu + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)dv} yY_s^\nu ds\Big|\mathcal{F}_t\Big]\Big)dt\Big], \qquad (2.6.16)$$

where they define the parameterized exponential local martingale Y_t^ν by

$$Y_t^\nu \triangleq \exp\Big\{-\int_0^t \Big(\theta_s^\top + \nu_s^\top\Big)dW_s - \frac{1}{2}\int_0^t \Big(\|\theta_s\|^2 + \|\nu_s\|^2\Big)ds\Big\}. \qquad (2.6.17)$$

Here, the Hilbert space $H(\sigma)$ is defined as

$$H(\sigma) \triangleq \Big\{\nu \in K(\sigma) : \mathbb{E}\Big[\int_0^T \| \nu_s \|^2 \, ds < \infty\Big]\Big\},$$

and the appropriate set $K(\sigma)$ for ν is generally given by

$$K(\sigma) \triangleq \Big\{\nu : \int_0^T \|\nu_t\|^2 dt < \infty, \ a.s. \text{ and } \sigma_t\nu_t = 0, \ \forall t \in [0,T], a.s.\Big\}.$$

See *Karatzas, Lehoczky, Shreve and Xu* [40] for the detail definition and arguments.

Contrary to the optimal consumption problem without habit formation, the previous authors acknowledged that the optimization problem (2.6.16) is generically more difficult since the dual functional becomes non-convex over the parameter process $\nu \in H(\sigma)$, and some new techniques in non-convex optimization is evidently needed. However, we want to point out that a fundamental reason behind the mathematical challenges of the dual problem is that it is not appropriate to formulate the dual functional over a family of exponential local martingales.

As we discussed in the previous section, the optimal dual solution lies in the closure of the linear transform of a family of supermartingales for general incomplete semimartingale financial market. Therefore, it is reasonably convincing that the dual optimizer Y_t^* to the optimization problem (2.6.16) may not be a exponential local martingale. Equivalently speaking, if we formulate the dual functional in the form of (2.6.16), then the set of local martingale deflators is generally too small to contain the dual optimizer.

On the other hand, the market completion argument by *Karatzas, Lehoczky, Shreve and Xu* [40] should work in general, and as people know how to solve the utility maximization problem with habit formation in the complete market, we should also be able to play the same trick and solve the path dependent optimization problem by defining the correct dual functional and imposing some restrictions on the fictitious stocks.

According to Lemma 2.6.4 and our main results Theorem 2.6.1 and 2.6.2, we should have the dual problem in the form

$$\tilde{v}(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\Big[\int_0^T V(t, yY_tK_t)\Big], \tag{2.6.18}$$

where we define

$$K_t = \Big(1 + \delta_t \int_t^T e^{\int_t^s (\delta_v - \alpha_v)dv} ds\Big), \tag{2.6.19}$$

and it is clear we can apply the convex duality in *Kramkov and Schachermayer* [48], [49], and the optimal solution Y_t^* happens to be the parameterized exponential local martingale if we assume all the market coefficients are bounded, see the proof of maximal elements of set $\mathcal{Y}(1)$ in Example 4.1 by *Karatzas and Žitković* [43]. These results can successfully resolve the open problem mentioned by *Detemple and Zapatero* [25] and *Egglezos and Karatzas* [26] in the incomplete market with Itô process models.

We end up this section with an explicit example and we consider the utility function given by $U(t,x) = \log(x)$, such that the conjugate utility function is $V(t,y) = -\log(y) - 1$.

We give the same construction of the financial market as in *Delbaen and Schachermayer* [22], see also example 5.1 in *Kramkov and Schachermayer* [48]. One the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by two independent Brownian motions B and W on $(\Omega, \mathbb{F}, \mathbb{P})$.

Define the process L by

$$L_t = \exp(B_t - \frac{1}{2}t), \quad t \geq 0.$$

Define the stopping time τ by

$$\tau = \inf\{t \geq 0 : L_t = \frac{1}{2}\}.$$

Clearly, we have $\tau < \infty$ a.s. Similarly, construct a martingale

$$M_t = \exp(W_t - \frac{1}{2}t).$$

The stopping time ι is defined as

$$\iota = \inf\{t \geq 0 : M_t = 2\}.$$

Define the financial market with the time horizon

$$T = \tau \wedge \iota,$$

and define the stock price process

$$S_t = \exp(-B_t + \frac{1}{2}t),$$

such that $b_t \equiv 1$ and $\sigma_t \equiv -1$ and the Bond price equals constant 1 at any time t for simplicity.

Theorem 2.1 in *Delbaen and Schachermayer* [22] showed the process Y defined as

$$Y_t^* = L_{\tau \wedge \iota \wedge t}$$

is a strictly local martingale under $\mathbb{P}$ and corresponds to Y_t^ν for $\nu \equiv 0$ by the definition of Y_t^ν in (2.6.17). And similar to the argument by Proposition 5.1 in *Kramkov and Schachermayer* [48], we can show that $Y_t^* K_t$ is the unique optimal solution of the dual optimization problem (2.6.18).

To this end, for each $Y \in \mathcal{Y}$, the process $\frac{YK}{Y^*K} = YS$ is a supermartingale starting at $Y_0 S_0 = 1$. Jensen's inequality implies that

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T \log\Big(Y_t K_t\Big) dt\Big] &= \mathbb{E}\Big[\int_0^T \log\Big(\frac{Y_t K_t}{Y_t^* K_t}\Big) dt\Big] + \mathbb{E}\Big[\int_0^T \log\Big(Y_t^* K_t\Big) dt\Big] \\
&\leq \int_0^T \log\Big(\mathbb{E}\Big[\frac{Y_t K_t}{Y_t^* K_t}\Big]\Big) dt + \mathbb{E}\Big[\int_0^T \log\Big(Y_t^* K_t\Big) dt\Big] \\
&\leq \mathbb{E}\Big[\int_0^T \log\Big(Y_t^* K_t\Big) dt\Big].
\end{aligned}$$

Also, following exactly the same proof of Proposition 5.1 in *Kramkov and Schachermayer* [48], we can show that $(Y_t^* M_{t\wedge T})_{t\geq 0}$ is the density process of an equivalent martingale measure and hence $\mathcal{M} \neq \emptyset$ and $\hat{v}(1) < \infty$. This completes the proof of our claim that $Y_t^* K_t$ is the unique optimal solution of the dual optimization problem (2.6.18).

Moreover, we can choose some special discounting processes δ_t and α_t such that there does not exist an exponential local martingale Y^ν such that

$$Y_t^* K_t = Y_t^\nu + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} Y_s^\nu ds \Big| \mathcal{F}_t\Big]. \tag{2.6.20}$$

To this end, we choose δ_t and α_t such that the process

$$G_t \triangleq Y_t^* K_t - \delta_t \mathbb{E}\Big[\int_t^T e^{-\int_t^s \alpha_v dv} Y_s^* K_s ds \Big| \mathcal{F}_t\Big]$$

is a continuous nonnegative semimartingale, this can be easily achieved by taking δ_t and K_t to be continuous semimartingales since the product of two semimartingales is still a semimartingale. Furthermore, we choose δ_t and α_t such that the finite variation process A_t appeared in the decomposition of $G_t = A_t + R_t$ is not identically the constant $1 - R_0$.

Then if (2.6.20) holds for some exponential local martingale Y^ν, it is easy to verify by Tonelli's theorem that

$$Y_t^\nu = G_t,$$

which contradicts the condition that the finite variation process is not identically $1 - R_0$, since a continuous local martingale with finite variation is a constant. This eventually provides us an counterexample to show the set of parameterized exponential local martingales is too small to contain the dual optimizer for the dual problem (2.6.16), however, it is the proper dual space for dual problem (2.6.18) and (2.6.18) is the correct dual problem to the associated utility maximization problem with consumption habit formation in the Itô process model.

Chapter 3

An Example of Utility Maximization with Consumption Habit Formation and Partial Observations

3.1 Introduction

This chapter aims to examine an example of the utility maximization problem with consumption habit formation in incomplete markets with partial observations, which can not be covered by the main results in Chapter 2. Specifically, we are interested in finding the optimal strategies for the habit forming investor in the incomplete market driven by Itô processes, together with the constraint of partial observations to the market randomness. We are facing the case that the individual investor develops his own consumption habits during the whole investment period and meanwhile has only access to the public stock price information $\mathbb{F}^S$. In other words, he/she can not observe the mean rate of return process μ_t and the corresponding Brownian motion B_t^1 which appears in the stock price dynamics. In our model, we will assume μ_t follows the mean reverting Ornstein Uhlenbeck process driven by another correlated Brownian motion B_t^2.

Our contributions are two-fold. 1). From the modeling perspective, we are investigating the utility maximization problem with consumption habit formation in the setting of incomplete financial markets driven by two Brownian motions B_t^1 and B_t^2 that are not perfectly correlated. Additionally, we impose the realistic incomplete information constraint on the individual investor. We assume he/she only has the access to the public stock prices, but can not observe the drift process appeared in the dynamics of the stock price process. The combination of these two scenarios is not only a novel framework and mathematically interesting, but also it covers many realistic constraints that the individual investor is facing in the daily life. 2). On the other hand, at the mathematical level, we solve the relatively complicated nonlinear HJB equation fully explicitly using some technical transformations. As a consequence, we furthermore derive the $\mathcal{F}_t^S$-adapted optimal investment and consumption polices in feedback form via rigorous verification arguments. Our analytical approach allows us to avoid proving the Dynamic Programming Principle and the measurable selection arguments associated with it.

Optimal investment problems under incomplete information have been studied by numerous authors, and we only list a very small subset of them: *Lakner* [52] applies martingale methods and derived the structure of the optimal investment strategies. The linear diffusion model is studied by *Brendle* [12], who derives explicit results for the value of information on optimal investment with power and exponential utilities using dynamic programming approach. The effects of learning on the composition of the optimal portfolios are studied in *Brennan* [13] and *Xia* [74]. By applying the duality approach, *Monoyios* [57] considers the optimal investment and hedging with both the uncertainty of the drift parameter and noisy knowledge at time 0 of the terminal value of the Brownian motion driving the stock. This problem eventually leads to a stochastic optimization problem under partial and inside information, and he obtains an explicit solution via Kalman-Bucy filtering together with techniques of enlargement of filtration. *Björk, Davis and Landén* [9] examine the market model with unobservable rates of returns that are allowed to be arbitrary semimartingales, and they provide a unified treatment for a large class of partially observed investment problems along with stochastic representations of the optimal terminal wealth and portfolio strategies.

To the best of our knowledge, the path dependent utility maximization under incomplete information is not yet addressed in the literature. However, we still want to single out two papers, *Munk* [58] and *Brendle* [12] which are technically close to our problem. *Munk* [58] tackles the utility maximization with consumption habit formation in the complete market, where he assumes the market price of risk process obeys a mean reverting SDE and he makes a strong assumption that the market price of risk is perfectly (negatively) correlated with the price level, i.e., his stock price process and the drift process can be taken as driven by the same Brownian Motion. By applying the Market Isomorphism result by *Schroder and Skiadas* [71] to the paper by *Wachter* [73], he obtains the closed form solution for the HJB equation and optimal control policies under power utility preference for only $p < 0$. Mathematically speaking, the solution for his HJB equation is very close to our final result. However, it is generally not realistic to assume there exists only one Brownian motion driving both the stock price process and its drift process. On the other hand, *Brendle* [12] treats the problem of utility maximization on the terminal wealth in the incomplete Itô process market under partial observations. He figures out the solution of the HJB equation can be expressed in a closed form by solving some ODE systems with time dependent parameters. Moreover, by setting a clever substitution, he proves that solving the previous ODEs is actually equivalent to solving a family of ODEs with constant parameters, which already have been discussed earlier by *Kim and Omberg* [45] for a different problem setting. *Kim and Omberg* [45] have solved these ODEs fully explicitly in various cases, which also assist us to obtain our explicit solutions depending on the market parameters, see Appendix A for the detail. However, when the intertemporal consumption choice comes into play, together with path dependent habit formation impact, it is not clear whether the HJB equation can still admit an explicit solution. Otherwise, the problem will become surprisingly difficult if we need to resort to the viscosity solution, as the solution of the HJB under partial observation filtration has five dimensions. Our notable contribution can also be summarized as that we combine the advantages of the two models considered by the previous authors, and successfully solve the *Munk*'s target problem in the *Brendle*'s incomplete market setting with partial observations. Furthermore, we provide the rigorous verification of the main results which is missing in the two previous economics papers.

The structure of the present chapter is outlined as: Section 3.2 introduces the market model and the concept of habit formation process. The utility maximization problem with addictive habit formation and partial observations is defined in Section 3.3. By applying the Kalman Bucy filtering theorem in Chapter 1 and dynamic programming arguments, we formally derive the Hamilton-Jacobi-Bellman(HJB) equation for the power utility preference and we provide the decoupled form solution of this nonlinear PDE. This reduces the algorithm to solving some auxiliary ODEs with constant coefficients. Based on these explicit smooth solutions, the explicit feedback form of the optimal investment and consumption policies will be obtained. Section 3.4 contains rigorous proofs of the corresponding verification arguments. At last, four cases of fully explicit solutions of some auxiliary ODEs are presented in the Appendix A.

3.2 Market Model and Consumption Habit Formation

On a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with the background filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, which satisfies the usual conditions, we consider a financial market with one risk-free bond and one stock account for a "small investor" over a finite time horizon $[0, T]$. The price of the bond S_t^0 solves:

$$dS_t^0 = r_t S_t^0 dt, \quad 0 \leq t \leq T$$

with initial price $S_0^0 = 1$, and without loss of generality, we assume the interest rate $r_t \equiv 0$, for all $t \in [0, T]$, this can be achieved by the standard change of numéraire.

The stock price S_t is modeled as a diffusion process solving:

$$dS_t = \mu_t S_t dt + \sigma_S S_t dB_t^1, \quad 0 \le t \le T, \tag{3.2.1}$$

with $S_0 = s > 0$, where the drift process μ is $(\mathcal{F}_t)_{0 \le t \le T}$ adapted, and satisfies the mean-reverting Ornstein Uhlenbeck SDE:

$$d\mu_t = -\lambda(\mu_t - \bar{\mu})dt + \sigma_\mu dB_t^2, \quad 0 \le t \le T. \tag{3.2.2}$$

Here, B^1 and B^2 are $(\mathcal{F}_t)_{0 \le t \le T}$ adapted Brownian motions defined on $(\Omega, \mathbb{F}, \mathbb{P})$ and they are correlated with the coefficient $\rho \in [-1, 1]$, i.e., $\langle B^1, B^2 \rangle_t = \rho t$. We assume the initial value of the drift process μ_0 is a $\mathcal{F}_0$ measurable Gaussian random variable, satisfying $\mu_0 \sim \mathrm{N}(\eta_0, \theta_0)$, which is independent of Brownian motions $(B_t^1)_{0 \le t \le T}$ and $(B_t^2)_{0 \le t \le T}$. We also assume all the coefficients $\sigma_S \ge 0, \lambda \ge 0, \bar{\mu}, \sigma_\mu \ge 0$ are constants.

Remark 3.2.1. *In this incomplete market under full background filtration $\mathbb{F}$, we do not assume the existence of the equivalent local martingale measures, for example, the exponential local martingale deflator well defined by*

$$H_t = \exp\Big(- \int_0^t \frac{\mu_v}{\sigma_S} dB_v^1 - \frac{1}{2} \int_0^t \frac{\mu_v^2}{\sigma_S^2} dv \Big), \quad 0 \le t \le T. \tag{3.2.3}$$

is allowed to be a strict local martingale. In the context of Karatzas and Kardaras [38], the existence of a local martingale deflator is proved to be equivalent to the condition of No Unbounded Profit with Bounded Risk (NUPBR), which is slightly weaker than the prevalent assumption to the market called No Free

Lunch with Vanishing Risk (NFLVR) defined by Delbaen and Schachermayer [20]. In other words, under the full observations information, we generally allow the failure of No Arbitrage condition.

Now, at each time $t \in [0,\ T]$, the investor chooses a consumption rate $c_t \geq 0$, and decides the amounts π_t of his/her wealth to invest in the stock, and the rest in bank. Then, in this self-financing market model, the investor's total wealth process X_t follows the dynamics:

$$dX_t = (\pi_t \mu_t - c_t)dt + \sigma_S \pi_t dB_t^1, \quad 0 \leq t \leq T, \tag{3.2.4}$$

with the initial wealth $X_0 = x_0 > 0$.

In this Chapter, we adopt the same definition of the habit formation process $Z_t \triangleq Z(t; (c_s)_{0 \leq s \leq t})$, which is also called "the standard of living" process, satisfying

$$dZ_t = (\delta(t)c_t - \alpha(t)Z_t)dt, \quad 0 \leq t \leq T, \tag{3.2.5}$$

where $Z_0 = z_0 \geq 0$ is called the *initial habit*, and $\alpha(t)$, $\delta(t)$ are now assumed to be nonnegative continuous functions.

Remark 3.2.2. *In this chapter, we assume the discounting factors are merely deterministic continuous functions in order to invoke the one-dimensional Kalman-Bucy filtering theorem as well as the Dynamic Programming Arguments.*

we shall resume the previous constraint on the consumption by the so called "addictive habit formation", i.e., we require investor's current consumption strategies shall never fall below the standard of living level,

$$c_t \geq Z_t, \quad \forall 0 \leq t \leq T, \quad a.s.. \tag{3.2.6}$$

We will see in the Chapter that this additional consumption budget constraint implies initial wealth must be sufficiently large to sustain habits and ensure the existence of optimal policies in a different point of view.

3.3 Utility Maximization with Kalman-Bucy Filtering

3.3.1 Dynamic Programming Arguments on Partial Observations Filtration $\mathbb{F}^S$

From now on, we shall make the assumption that the investor can observe the stock price process S_t which is published and available to the public, however, the drift process μ_t and the information of Brownian motions $(B_t^1)_{0\leq t\leq T}$ and $(B_t^2)_{0\leq t\leq T}$ are unknown. We shall call this as the "*partial observations information*" scenario. This investment and consumption optimization problem with incomplete information will be modeled by requiring the investment strategy $(\pi_t)_{0\leq t\leq T}$ and consumption policy $(c_t)_{0\leq t\leq T}$ be only adapted to the partial observation filtration $\mathbb{F}^S = (\mathcal{F}_t^S)_{0\leq t\leq T}$ where $\mathcal{F}_t^S = \sigma\{S_u : 0 \leq u \leq t\}$, which is strictly smaller than the background full information $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$.

Applying the previous Kalman-Bucy Stochastic Filtering Theorem introduced in Chapter 1, we can first define the **Innovation Process** in our market model as:

$$d\hat{B}_t \triangleq \frac{1}{\sigma_S}\Big[(\mu_t - \hat{\mu}_t)dt + \sigma_S dB_t^1\Big] = \frac{1}{\sigma_S}\Big(\frac{dS_t}{S_t} - \hat{\mu}_t dt\Big), \quad 0 \leq t \leq T, \tag{3.3.1}$$

which is a Brownian motion under partial observations filtration $\mathcal{F}_t^S$, where the process $\hat{\mu}_t = \mathbb{E}\Big[\mu_t \Big| \mathcal{F}_t^S\Big]$ is the conditional estimation of drift process μ_t.

Moreover, by the same Kalman-Bucy filtering theorem, the process $\hat{\mu}_t$ satisfies the linear SDE:

$$\begin{aligned} d\hat{\mu}_t =& -\lambda(\hat{\mu}_t - \bar{\mu})dt + \Big(\frac{\hat{\Omega}_t + \sigma_S\sigma_\mu\rho}{\sigma_S}\Big)d\hat{B}_t, \\ =& \Big(-\lambda - \frac{\Big(\hat{\Omega}_t + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\Big)\hat{\mu}_t dt + \lambda\bar{\mu}dt + \frac{\Big(\hat{\Omega}_t + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{dS_t}{S_t}, \quad 0 \le t \le T, \end{aligned} \tag{3.3.2}$$

with $\hat{\mu}_0 = \mathbb{E}\Big[\mu_0 \Big| \mathcal{F}_0^S\Big] = \eta_0$. So we can solve for $\hat{\mu}_t$ as the strong solution of the SDE (3.3.2) by knowing the stock price process S_t and $\hat{\Omega}_t$.

And the conditional variance $\hat{\Omega}_t = \mathbb{E}\Big[(\mu_t - \hat{\mu}_t)^2 \Big| \mathcal{F}_t^S\Big]$ satisfies the deterministic Riccati ODE:

$$d\hat{\Omega}_t = \Big[-\frac{1}{\sigma_S^2}\hat{\Omega}_t^2 + \Big(-\frac{2\sigma_\mu\rho}{\sigma_S} - 2\lambda\Big)\hat{\Omega}_t + (1-\rho^2)\sigma_\mu^2\Big]dt, \quad 0 \le t \le T, \tag{3.3.3}$$

with $\hat{\Omega}(0) = \mathbb{E}\Big[(\mu_0 - \eta)^2 \Big| \mathcal{F}_0^S\Big] = \theta_0$, which has an explicit solution as:

$$\hat{\Omega}_t = \hat{\Omega}(t;\theta_0) = \sqrt{k}\sigma_S \frac{k_1 \exp(2(\frac{\sqrt{k}}{\sigma_S})t) + k_2}{k_1 \exp(2(\frac{\sqrt{k}}{\sigma_S})t) - k_2} - (\lambda + \frac{\sigma_\mu\rho}{\sigma_S})\sigma_S^2, \quad 0 \le t \le T, \tag{3.3.4}$$

where:

$$\begin{aligned} k &= \lambda^2\sigma_S^2 S + 2\sigma_S\sigma_\mu\lambda\rho + \sigma_\mu^2, \\ k_1 &= \sqrt{k}\sigma_S + (\lambda\sigma_S^2 + \sigma_S\sigma_\mu\rho) + \theta_0, \\ k_2 &= -\sqrt{k}\sigma_S + (\lambda\sigma_S^2 + \sigma_S\sigma_\mu\rho) + \theta_0. \end{aligned}$$

By simple observation, we see $\hat{\Omega}(t)$ converges monotonically to the value

$$\theta^* = \sigma_S\sqrt{\lambda^2\sigma_S^2 + 2\sigma_S\sigma_\mu\lambda\rho + \sigma_\mu^2} - (\lambda\sigma_S^2 + \sigma_S\sigma_\mu\rho) > 0 \tag{3.3.5}$$

as time $t \to +\infty$, which we call as "steady state learning" (see also *Brennan* [13]). This convergence property of $\hat{\Omega}(t)$ tells us the precision of the drift estimate goes from an initial condition to a steady state in the long time run, and after large time T, new return observations contribute to updating the estimated value of the state variable, but seldom reduce the variance of the estimation error. More precisely, by the evolution of Riccati ODE (3.3.3), we have the monotone solution $\hat{\Omega}(t)$ on $(0, \infty)$ has the bounds

$$\min(\theta_0, \theta^*) \leq \hat{\Omega}(t) \leq \max(\theta_0, \theta^*), \quad \forall t \geq 0. \tag{3.3.6}$$

Under the observation filtration $(\mathcal{F}_t^S)_{0 \leq t \leq T}$, we can instead rewrite stock price dynamics (3.2.1) driven by the innovation process $\hat{W}_t$ as:

$$dS_t = \hat{\mu}_t S_t dt + \sigma_S S_t d\hat{B}_t, \quad 0 \leq t \leq T. \tag{3.3.7}$$

Remark 3.3.1. *We now revisit Remark* 3.2.1*. Although the original financial market is incomplete under the full background filtration $\mathbb{F}$, possibly with slight arbitrage opportunities, when we are restricted to the partial observation filtration $\mathbb{F}^S = (\mathcal{F}_t^S)_{0 \leq t \leq T}$, the market becomes complete together with NFLVR condition in the investor's point of view. To wit, our price process and conditional drift process are both driven by the same Brownian motion $\hat{W}$ under $\mathbb{F}^S$, moreover, we will show later in Proposition* 3.4.1 *that there also exists a equivalent local martingale measure under $\mathbb{F}^S$. These facts are essential for the derivation of the decoupled form solution of the associated PDE as well as the proof of its verification arguments.*

Notice we are now seeking the optimal investment and consumption strategies π_t and c_t which are only progressively measurable with respect to the partial observations filtration $\mathcal{F}_t^S$, where the living standard process Z_t satisfies the ODE

$$dZ_t = (\delta(t)c_t - \alpha(t)Z_t)dt, \quad 0 \le t \le T, \tag{3.3.8}$$

and under the partial observations filtration $\mathcal{F}_t^S$, the wealth process dynamics (3.2.4) under π_t and c_t will be rewritten as:

$$dX_t = (\pi_t \hat{\mu}_t - c_t)dt + \sigma_S \pi_t d\hat{B}_t, \quad 0 \le t \le T. \tag{3.3.9}$$

Our goal now is to maximize the consumption with linear habit formation and terminal wealth by power utility preference under the partial observations filtration $\mathcal{F}_t^S$:

$$v(x_0, z_0, \eta_0, \theta_0) = \sup_{\pi, c \in \mathcal{A}} \mathbb{E}\Big[\int_0^T \frac{(c_s - Z_s)^p}{p} ds + \frac{(X_T)^p}{p}\Big], \tag{3.3.10}$$

where we take the risk aversion coefficient $p < 1$ and $p \neq 0$.

Our aim is to provide an analytic solution of the control problem (3.3.10) using direct dynamic programming, i.e., first solve the Dynamic Programming Equation analytically and then perform a rigorous verification argument. Therefore, there is no need to either define the value function at later times or to prove the Dynamic Programming Principle involving some complicated measurable selection arguments.

Formally, at this level, we are now looking for a smooth function $\tilde{v}(t, x, z, \eta, \theta)$

defined on an appropriate domain such that the process

$$\int_0^t \frac{(c_s - Z_s)^p}{p} ds + \tilde{v}(t, X_t, Z_t, \hat{\mu}_t, \hat{\Omega}_t), \quad \forall t \in [0, T],$$

is a local supermartingale for each admissible control $(\pi_t, c_t) \in \mathcal{A}$ and a local martingale for the optimal feedback control $(\pi_t^*, c_t^*) \in \mathcal{A}$. The appropriate domain will be carefully defined later after we solve the associated HJB equation explicitly, moreover, some financial intuitions will also be clarified based on the domain of the solution.

Furthermore, we recall that the conditional variance process $\hat{\Omega}_t = \hat{\Omega}(t, \theta_0)$ is actually a deterministic function of time explicitly given by (3.3.4). We can therefore set the variable θ in the definition of $\hat{v}$ by a deterministic function $\theta = \theta(t, \theta_0)$ depending on the parameter θ_0 to reduce the dimension of the function $\tilde{v}$, i.e., the variable $\theta(t; \theta_0)$ is absorbed by the variable t. Hence, we can define the function $V(t, x, z, \eta; \theta_0)$ as

$$V(t, x, z, \eta; \theta_0) \triangleq \tilde{v}(t, x, z, \eta, \theta(t, \theta_0)),$$

and our target above can be simplified into finding a smooth enough function $V(t, x, z, \eta; \theta_0)$ on some appropriate domain , denoted by $V(t, x, z, \eta)$ for simplicity, such that

$$\int_0^t \frac{(c_s - Z_s)^p}{p} ds + V(t, X_t, Z_t, \hat{\mu}_t), \quad \forall t \in [0, T],$$

is a local supermartingale for each admissible control $(\pi_t, c_t) \in \mathcal{A}$ and a local martingale for the optimal feedback control $(\pi_t^*, c_t^*) \in \mathcal{A}$, for each fixed initial value $\hat{\Omega}(0) = \theta_0$.

An investment and consumption pair process (π_t, c_t) is said in the **Admissible Control Space** $\mathcal{A}$: if it is $\mathcal{F}_t^S$-progressively measurable, and satisfies the integrability conditions:

$$\int_0^T \pi_t^2 dt < +\infty, \quad a.s. \text{ and } \int_0^T c_t dt < +\infty, \quad a.s. \tag{3.3.11}$$

with the addictive habits constraint that: $c_t \geq Z_t, \quad \forall t \in [0, T]$. Moreover, no bankruptcy is allowed, i.e., the investor's wealth remains nonnegative: $X_t \geq 0,\ 0 \leq t \leq T$.

By the definition of $V(t, x, z, \eta)$ and Ito's formula, we can formally derive the Hamilton-Jacobi-Bellman (HJB) equation as:

$$\begin{aligned} & V_t - \alpha(t) z V_z - \lambda(\eta - \bar{\mu}) V_\eta + \frac{\left(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\right)^2}{2\sigma_S^2} V_{\eta\eta} + \max_{c \in \mathcal{A}} \Big[- c V_x + c\delta(t) V_z \\ & + \frac{(c-z)^p}{p} \Big] + \max_{\pi \in \mathcal{A}} \Big[\pi \eta V_x + \frac{1}{2} \sigma_S^2 \pi^2 V_{xx} + V_{x\eta} \Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho \Big) \pi \Big] = 0, \end{aligned} \tag{3.3.12}$$

with the terminal condition $V(T, x, z, \eta) = \frac{x^p}{p}$.

3.3.2 The Decoupled Reduced Form Solution

If $V(t, x, z, \eta)$ is smooth enough, the first order condition formally derives

$$\begin{aligned} \pi^*(t, x, z, \eta) &= \frac{-\eta V_x - \left(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\right) V_{x\eta}}{\sigma_S^2 V_{xx}}, \\ c^*(t, x, z, \eta) &= z + \Big(V_x - \delta(t) V_z \Big)^{\frac{1}{p-1}}. \end{aligned} \tag{3.3.13}$$

which achieve the maximum over control policies π and c respectively.

Plugging forms of (3.3.13) for π^* and c^*, the HJB equation becomes:

$$V_t - \alpha(t)zV_z - \lambda(\eta - \bar{\mu})V_\eta + \frac{(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho)^2}{2\sigma_S^2}V_{\eta\eta} - \frac{\eta(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho)}{\sigma_S^2}\frac{V_x V_{x\eta}}{V_{xx}}$$
$$-\frac{\eta^2}{2\sigma_S^2}\frac{V_x^2}{V_{xx}} - \frac{(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho)^2}{2\sigma_S^2}\frac{V_{x\eta}^2}{V_{xx}} - z(V_x - \delta(t)V_z) - \frac{p-1}{p}\Big(V_x - \delta(t)V_z\Big)^{\frac{p}{p-1}} = 0. \tag{3.3.14}$$

We expect that the smooth solution $V(t, x, z, \eta)$ of the HJB equation at time $t = 0$ is actually the value function, i.e., $V(0, x_0, z_0, \eta_0; \theta_0) = v(x_0, z_0, \eta_0, \theta_0)$. Due to the homogeneity property of the power utility function and the linearity of dynamics (3.3.9) and (3.3.8) for X_t and Z_t respectively, it's easy to see that if $V(t, x, z, \eta)$ is finite, then it is homogeneous in (x, z) with degree p, i.e., for any $x > 0$, $z \geq 0$ and the positive constant k, we have $V(t, kx, kz, \eta) = k^p V(t, x, z, \eta)$. It therefore makes sense for us to seek the value function of the form:

$$V(t, x, z, \eta) = \frac{\Big[(x - W(t, \eta)z)\Big]^p}{p}M(t, \eta)$$

for some test functions $W(t, \eta)$ and $M(t, \eta)$ to be determined. By the virtue of $V(T) = \frac{x^p}{p}$, we will require $M(T, \eta) = 1$ and $W(T, \eta) = 0$.

After we do the direct substitution in the above Equation (3.3.14) and divide the equation on both sides by $(x-W(t,\eta)z)^p$, the HJB equation becomes

$$\frac{\Big[f(t,W)z+\lambda(\eta-\bar{\mu})W_\eta-\frac{1}{2\sigma_S^2}(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2W_{\eta\eta}+\frac{\eta}{\sigma_S^2}(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)W_\eta\Big]}{x-W(t,\eta)z}M$$
$$+\frac{1}{p}M_t-\frac{\lambda(\eta-\bar{\mu})}{p}M_\eta+\frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{2p\sigma_S^2}M_{\eta\eta}-\frac{\eta(\hat{\Omega}(t)+\sigma_S\sigma_S\rho)}{(p-1)\sigma_S^2}M_\eta$$
$$-\frac{\eta^2}{2(p-1)\sigma_S^2}M-\frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{2(p-1)\sigma_S^2}\frac{M_\eta^2}{M}-\frac{p-1}{p}\Big(1+\delta(t)W(t,\eta)\Big)^{\frac{p}{p-1}}M^{\frac{p}{p-1}}=0. \tag{3.3.15}$$

where we set

$$f(t,W)=-W_t+\alpha(t)W-(1+\delta(t)W).$$

Since the Equation (3.3.15) above holds for all values of x and z, we can naturally set the unknown priori function $W(t,\eta)=W(t)$ as a deterministic function in time t and independent of the variable η which satisfies:

$$f(t,W)=-W_t(t)+\alpha(t)W(t)-(1+\delta(t)W(t))=0 \tag{3.3.16}$$

with the terminal condition $W(T)=0$, which is equivalent to:

$$W(t)=\int_t^T\exp\Big(\int_t^s(\delta(v)-\alpha(v))dv\Big)ds. \quad 0\le t\le T. \tag{3.3.17}$$

Remark 3.3.2. *We will discuss later in Proposition* 3.4.1 *the hidden reason that why we can actually reduce the burden of dependence on η or θ_0 for function $W(t)$. This decoupled form does not hold for the utility maximization problem under full observations when the market is assumed to be incomplete and we do not require the existence of equivalent local martingale measures.*

Now we can substitute the function $W(t)$ into the equation (3.3.15) above, and simplify it as:

$$M_t + \frac{p\eta^2}{2(1-p)\sigma_1^2}M + \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}M_{\eta\eta} + (1-p)\Big(1+\delta(t)W(t)\Big)^{\frac{p}{p-1}}M^{\frac{p}{p-1}}$$
$$+\Big[-\lambda(\eta-\bar{\mu}) + \frac{\eta(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)p}{(1-p)\sigma_S^2}\Big]M_\eta + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2 p}{2(1-p)\sigma_S^2}\frac{M_\eta^2}{M} = 0. \tag{3.3.18}$$

Now in order to solve the above nonlinear PDE (3.3.18), we can set the power transform as

$$M(t,\eta) = N(t,\eta)^{1-p} \tag{3.3.19}$$

This idea of power transform was first introduced in *Zariphopoulou* [75].

And the nonlinear PDE (3.3.18) for $M(t,\eta)$ reduces to the linear parabolic PDE for $N(t,\eta)$ as:

$$N_t + \frac{p\eta^2}{2(1-p)^2\sigma_S^2}N(t,\eta) + \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}N_{\eta\eta} + \Big(1+\delta(t)W(t)\Big)^{\frac{p}{p-1}}$$
$$+\Big[-\lambda(\eta-\bar{\mu}) + \frac{\eta\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)p}{(1-p)\sigma_S^2}\Big]N_\eta(t,\eta) = 0 \tag{3.3.20}$$

with $N(T,\eta) = 1$.

For the above linear PDE (3.3.20) of $N(t,\eta)$, we can further solve it explicitly as:

$$N(t,\eta) = \int_t^T \Big(1+\delta(s)W(s)\Big)^{\frac{p}{p-1}} \exp\Big(A(s,t)\eta^2 + B(s,t)\eta + C(s,t)\Big)ds$$
$$+ \exp\Big(A(T,t)\eta^2 + B(T,t)\eta + C(T,t)\Big), \tag{3.3.21}$$

where we have for $0 \le t \le s \le T$, $A(s,t) = A(t;s)$, $B(s,t) = B(t;s)$ and $C(s,t) = C(t;s)$ satisfying the following ODEs:

$$A_t(t) + \frac{p}{2(1-p)^2\sigma_S^2} + 2\Big[-\lambda + \frac{p(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)}{\sigma_S^2(1-p)}\Big]A(t) + \frac{2(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}A^2(t) = 0; \tag{3.3.22}$$

$$B_t(t) + \Big[-\lambda + \frac{p(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)}{\sigma_S^2(1-p)}\Big]B(t) + 2\lambda\bar{\mu}A(t) + \frac{2(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}A(t)B(t) = 0; \tag{3.3.23}$$

$$C_t(t) + \lambda\bar{\mu}B(t) + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}\Big(B^2(t) + 2A(t)\Big) = 0; \tag{3.3.24}$$

with terminal conditions: $A(s) = B(s) = C(s) = 0$.

We remark that the above ODEs are similar to the ODEs obtained by *Brendle* [12] for terminal wealth optimization problem with partial observations, and he made an insightful observation that we can actually solve the above 3 ODEs with time t dependent coefficients by solving the following 5 auxiliary ODEs with constant coefficients, see section 4 of *Brendle* [12] for the detail proof.

Theorem 3.3.1. *For $0 \le t \le s \le T$, consider the following auxiliary ODEs for $a(t)$, $b(t)$, $c(t)$, $f(t)$ and $g(t)$:*

$$a_t(t) = -\frac{2(1-p+p\rho^2)}{1-p}\sigma_\mu^2 a^2(t) + \Big(2\lambda - \frac{2p\rho\sigma_\mu}{(1-p)\sigma_S}\Big)a(t) - \frac{p}{2(1-p)\sigma_S^2}, \tag{3.3.25}$$

$$b_t(t) = -\frac{2(1-p+p\rho^2)}{1-p}\sigma_\mu^2 a(t)b(t) - 2\lambda\bar{\mu}a(t) + \Big(\lambda - \frac{p\rho\sigma_\mu}{(1-p)\sigma_S}\Big)b(t), \quad (3.3.26)$$

$$c_t(t) = -\sigma_\mu^2 a(t) - \frac{(1-p+p\rho^2)\sigma_\mu^2}{2(1-p)}b^2(t) - \lambda\bar{\mu}b(t), \quad (3.3.27)$$

$$f_t(t) = -2(1-\rho^2)\sigma_\mu^2 f^2(t) + 2\frac{\lambda\sigma_S + \rho\sigma_\mu}{\sigma_S}f(t) + \frac{1}{2\sigma_S^2}, \quad (3.3.28)$$

$$g_t(t) = \sigma_\mu^2(1-\rho^2)(f(t) - a(t)), \quad (3.3.29)$$

with the terminal conditions $a(s) = b(s) = c(s) = f(s) = g(s) = 0$*, and if we adopt the convention* $\frac{0}{0} = 0$*, then for the functions defined by:*

$$\tilde{A}(t;s) = \frac{a(t)}{(1-p)\Big(1 - 2a(t)\hat{\Omega}(t)\Big)},$$

$$\tilde{B}(t;s) = \frac{b(t)}{(1-p)\Big(1 - 2a(t)\hat{\Omega}(t)\Big)},$$

$$\begin{aligned}\tilde{C}(t;s) =& \frac{1}{1-p}\Big[c(t) + \frac{\hat{\Omega}(t)}{\Big(1 - 2a(t)\hat{\Omega}(t)\Big)}b^2(t) - \frac{1-p}{2}\log\Big(1 - 2a(t)\hat{\Omega}(t)\Big) \\ & - \frac{p}{2}\log\Big(1 - 2f(t)\hat{\Omega}(t)\Big) - pg(t)\Big],\end{aligned}$$

we have the equivalence that:

$$A(t;s) = \tilde{A}(t;s),\ B(t;s) = \tilde{B}(t;s),\ C(t;s) = \tilde{C}(t;s), \quad 0 \le t \le s \le T. \quad (3.3.30)$$

Remark 3.3.3. *The equivalence result* (3.3.30) *reveals that solving the ODEs* (3.3.22)*,* (3.3.23)*,* (3.3.24) *with variable coefficients is equivalent to solving the auxiliary ODEs* (3.3.25)*,* (3.3.26)*,* (3.3.27)*,* (3.3.28) *and* (3.3.29) *with constant coefficients in such an order that we solve the Riccati ODE* (3.3.25) *first,* *and substitute the solution $a(t;s)$ into ODE* (3.3.26) *and solve for the solution* $b(t;s)$*, and etc.*

Actually, we can even solve out fully explicit solutions for $a(t;s)$*,* $b(t;s)$*,* $c(t;s)$*,* $f(t;s)$ *and* $g(t;s)$*. We list all four different cases of fully explicit solutions in the Appendix depending on the risk aversion coefficient p and the market coefficients σ_S, σ_μ, λ and ρ. By simple substitutions, we can therefore solve the ODEs* (3.3.22)*,* (3.3.23)*,* (3.3.24) *for* $A(t;s)$*,* $B(t;s)$ *and* $C(t;s)$ *fully explicitly.*

Lemma 3.3.2. *Suppose the risk aversion constant p and the market coefficients σ_S, σ_μ, λ, ρ satisfy*

$$\frac{p}{1-p} \leq \frac{\lambda^2\sigma_S^2}{(2\lambda\rho\sigma_\mu + \sigma_\mu^2)} \leq \frac{\lambda\sigma_S}{\rho\sigma_\mu}, \tag{3.3.31}$$

and the solution of (3.3.25) *satisfies* $|1 - a(t;s)\hat{\Omega}(t)| \geq \epsilon > 0$ *for a constant* ϵ *on* $0 \leq t \leq s \leq T$*, then there exist uniform constants bounds* $\bar{K}_1 > 0$*,* $\bar{K}_2 > 0$ *and* $\bar{K}_3 > 0$ *such that*

$$A(t;s) \leq \bar{K}_1, \quad B(t;s) \leq \bar{K}_2, \quad C(t;s) \leq \bar{K}_3, \quad 0 \leq t \leq s \leq T. \tag{3.3.32}$$

Proof. Under Assumption (3.3.31), the explicit solution $a(t;s)$ is bounded and we observe the form of ODEs (3.3.26), (3.3.27), then $b(t;s)$, $c(t;s)$ are bounded on $0 \leq t \leq s \leq T$ if $a(t;s)$ is bounded. Now since the ODE (3.3.28) is well defined independent of the risk aversion constant p, and it always admits a bounded solution $f(t;s) < 0$ and $|1 - f(t;s)\hat{\Omega}(t)| > 1 > 0$ for $0 \leq t \leq s \leq T$ and hence we deduce $g(t;s)$ is bounded for $0 \leq t \leq s \leq T$. Combine these with the assumption $|1 - a(t;s)\hat{\Omega}(t)| \geq \epsilon > 0$ on $0 \leq t \leq s \leq T$, we can conclude $A(t;s)$, $B(t;s)$ and $C(t;s)$ are all uniformly bounded on $0 \leq t \leq s \leq T$ by the equivalence results in Theorem 3.3.1. $\square$

Now, for $t \in [0,T]$, $\eta \in (-\infty, +\infty)$, we can define the *effective domain* for the pair (x,z) as:

$$(x,z) \in \mathbb{D}_t = \{(x', z') \in (0,+\infty) \times [0,+\infty);\ x' \geq W(t)z'\},\ 0 \leq t \leq T, \tag{3.3.33}$$

and the function

$$\begin{aligned} V(t,x,z,\eta) =& \Big[\int_t^T \Big(1 + \delta(s)W(s)\Big)^{\frac{p}{p-1}} \exp\Big(A(s,t)\eta^2 + B(s,t)\eta + C(s,t)\Big) ds \\ & + \exp\Big(A(T,t)\eta^2 + B(T,t)\eta + C(T,t)\Big)\Big]^{1-p} \frac{\Big[x - W(t)z\Big]^p}{p} \end{aligned} \tag{3.3.34}$$

is well defined on $[0,T] \times \mathbb{D}_t \times \mathbb{R}$ and it's the classical solution of the HJB equation (3.3.12), where $W(t) = \int_t^T \exp(\int_t^s (\delta(v) - \alpha(v))dv)ds$, and $A(s,t)$, $B(s,t)$, $C(s,t)$ are defined in (3.3.22), (3.3.23), (3.3.24).

Remark 3.3.4. *In our main result below, we want to verify that the above classical solution $V(t,x,z,\eta)$ at time $t = 0$ equals our primal value function defined in* (3.3.10)*, i.e., $V(0, x_0, z_0, \eta_0; \theta_0) = v(x_0, z_0, \eta_0, \theta_0)$. However, the effective domain of $V(t,x,z,\eta)$ motivates some constraints on the optimal wealth process X_t^* and habit formation process Z_t^*. To wit, function $V(t,x,z,\eta) = -\infty$ when $x < W(t)z$, which mandates that $X_t^* \geq W(t)Z_t^*$ for each $t \in [0,T]$ to ensure the process $V(t, X_t^*, Z_t^*, \hat{\mu}_t)$ is well defined. In particular, when $t = 0$, we have to mandate the initial wealth-habit budget constraint that $x_0 > W(0)z_0$.*

3.3.3 The Main Result

Theorem 3.3.3 (The Verification Theorem).

Build upon the initial wealth-habit budget constraint $x_0 > W(0)z_0$*, then either if risk aversion constant*

$$p < 0;$$

or if

$$0 < p < 1, \qquad \textit{together with market coefficients } \sigma_S, \sigma_\mu, \lambda, \Theta, \rho$$

satisfy the additional assumption (3.3.31) *and*

$$\frac{p(1+p)}{(1-p)^2} < \frac{\lambda^2 \sigma_S^4}{4(\Theta + \sigma_S \sigma_\mu \rho)^2}, \tag{3.3.35}$$

where $\Theta \triangleq \max\{\theta,\ \theta^*\}$ *and* θ^* *is define in* (3.3.5)*, moreover, we assume the upper bound* $\bar{K}_1$ *in* (3.3.32) *of* $A(t;s)$ *in Lemma* 3.3.2 *satisfies*

$$4\bar{K}_1 < \frac{\lambda \sigma_S^2}{(\Theta + \sigma_S \sigma_\mu \rho)^2}. \tag{3.3.36}$$

Then, the solution (3.3.34) *of HJB equation equals the value function defined in* (3.3.10)*:*

$$V(0, x_0, z_0, \eta_0; \theta_0) = v(x_0, z_0, \eta_0, \theta_0). \tag{3.3.37}$$

And the optimal investment policy π_t^* *and optimal consumption policy* c_t^* *are given in the feedback form:* $\pi_t^* = \pi^*(t, X_t^*, Z_t^*, \hat{\mu}_t)$ *and* $c_t^* = c^*(t, X_t^*, Z_t^*, \hat{\mu}_t)$*,* $0 \leq t \leq T$*, where the function* $\pi^*(t, x, z, \eta) : [0, T] \times \mathbb{D}_t \times \mathbb{R} \longrightarrow \mathbb{R}$ *is defined* *by:*

$$\pi^*(t,x,z,\eta) = \Big[\frac{\eta}{(1-p)\sigma_S^2} + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{N_\eta(t,\eta)}{N(t,\eta)}\Big](x - W(t)z),\ 0 \le t \le T. \tag{3.3.38}$$

$c^*(t,x,z,\eta) : [0,T] \times \mathbb{D}_t \times \mathbb{R} \longrightarrow \mathbb{R}_+$ *is defined by:*

$$c^*(t,x,z,\eta) = z + \frac{(x - W(t)z)}{\Big(1+\delta(t)W(t)\Big)^{\frac{1}{1-p}}N(t,\eta)}, \quad 0 \le t \le T. \tag{3.3.39}$$

And the optimal wealth process X_t^*, for $0 \le t \le T$, is given explicitly by:

$$X_t^* = (x_0 - W(0)z_0)\frac{N(t,\hat{\mu}_t)}{N(0,\eta)}\exp\Big(\int_0^t \frac{(\hat{\mu}_u)^2}{2(1-p)\sigma_S^2}du + \int_0^t \frac{\hat{\mu}_u}{(1-p)\sigma_S}d\hat{B}_u\Big) + W(t)Z_t^*, \tag{3.3.40}$$

where $W(t)$ and $N(t,\eta)$ are defined in (3.3.17) *and* (3.3.21) *respectively.*

Remark 3.3.5. *The more complex structure of feedback forms of optimal investment and consumption policies is the consequence of the time non-separability of the instantaneous utility with habit formation. We can see the portfolio/wealth ratio $\frac{\pi^*}{X^*}$ and consumption/wealth ratio $\frac{c^*}{X^*}$ are now depending on the habit-formation/wealth ratio $\frac{Z^*}{X^*}$:*

$$\frac{\pi^*}{X^*} = \Big[\frac{\hat{\mu}_t}{(1-p)\sigma_S^2} + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{N_\eta(t,\hat{\mu}_t)}{N(t,\hat{\mu}_t)}\Big](1 - W(t)\frac{Z^*}{X^*}),$$

and

$$\frac{c^*}{X^*} = \frac{1}{\Big(1+\delta(t)W(t)\Big)^{\frac{1}{1-p}}N(t,\hat{\mu}_t)} + \Big(1 - \frac{W(t)}{\Big(1+\delta(t)W(t)\Big)^{\frac{1}{1-p}}N(t,\hat{\mu}_t)}\Big)\frac{Z^*}{X^*},$$

Moreover, although function $c^*(\cdot, x, \cdot, \cdot)$ *remains still linear and increasing in* $x > 0$, $c^*(\cdot, \cdot, z, \cdot)$ *is not necessary increasing in* $z \geq 0$, *which shows the increase of initial habit dose not necessarily imply the increase of optimal consumption stream. And since the dependence of* $c^*(t, x, z, \eta)$ *on the discounting factors* $\alpha(t)$ *and* $\delta(t)$ *are even more complicated, the optimal consumption process* c_t^* *is not necessarily monotone in the habit formation process* Z_t^*.

3.4 Proof of The Verification Theorem

We will first show the consumption constraint $c_t \geq Z_t$ implies the constraint on the controlled wealth process by the following proposition:

Proposition 3.4.1. *The admissible space* $\mathcal{A}$ *is not empty if and only if the initial budget constraint with habit formation* $x_0 \geq W(0)z_0$ *is fulfilled. Moreover, for each pair of investment and consumption policy* $(\pi, c) \in \mathcal{A}$, *the controlled wealth process* $X_t^{\pi,c}$ *satisfies the constraint:*

$$X_t^{\pi,c} \geq W(t)Z_t, \quad 0 \leq t \leq T, \tag{3.4.1}$$

where the deterministic function $W(t)$ *is defined in* (3.3.16) *and refers the cost of subsistence consumption per unit of standard of living at time* t.

Proof. On one hand, let's assume $x_0 \geq W(0)z_0$, then we can always take $\pi_t \equiv 0$, and $c_t = z_0 \exp\Big(\int_0^t (\delta(v) - \alpha(v))dv\Big)$ for $t \in [0, T]$, it is easy to verify $X_t^{\pi,c} \geq 0$ and $c_t \equiv Z_t$ so that $(\pi, c) \in \mathcal{A}$, and hence $\mathcal{A}$ is not empty.

On the other hand, starting from $t = 0$ with the wealth x_0 and the standard of living z_0, the addictive habits constraint $c_t \geq Z_t,\ 0 \leq t \leq T$ implies the consumption must always exceed the *subsistence consumption* $\bar{c}_t = Z(t; \bar{c}_t)$ which satisfies

$$d\bar{c}_t = (\delta(t) - \alpha(t))\bar{c}_t dt, \quad \bar{c}_0 = z_0, \quad 0 \leq t \leq T, \tag{3.4.2}$$

Indeed, we first recall by the definition of Z_t that $dZ_t = (\delta_t c_t - \alpha_t Z_t)dt$ with $Z_0 = z \geq 0$, and the constraint that $c_t \geq Z_t$ implies

$$dZ_t \geq (\delta_t Z_t - \alpha_t Z_t)dt, \quad Z_0 = z_0. \tag{3.4.3}$$

By the simple subtraction of (3.4.3) and (2.3.8), one can get

$$d(Z_t - \bar{c}_t) \geq (\delta_t - \alpha_t)(Z_t - \bar{c}_t)dt, \quad Z_0 - \bar{c}_0 = 0,$$

from which we can derive that

$$e^{\int_0^t (\delta_s - \alpha_s)ds}(Z_t - \bar{c}_t) \geq 0, \quad \forall t \in [0, T]. \tag{3.4.4}$$

And hence we can obtain $c_t \geq \bar{c}_t$, which is equivalent to

$$c_t \geq z_0 \exp\Big(\int_0^t (\delta(v) - \alpha(v))dv \Big), \quad 0 \leq t \leq T. \tag{3.4.5}$$

Define the exponential local martingale

$$\widetilde{H}_t = \exp\Big(- \int_0^t \frac{\hat{\mu}_v}{\sigma_S} d\hat{B}_v - \frac{1}{2} \int_0^t \frac{\hat{\mu}_v^2}{\sigma_S^2} dv \Big), \quad 0 \leq t \leq T. \tag{3.4.6}$$

Since $\hat{\mu}_t$ follows the dynamics (3.3.2), which is

$$\hat{\mu}_t = e^{-t\lambda}\eta + \bar{\mu}(1 - e^{-t\lambda}) + \int_0^t e^{\lambda(u-t)} \frac{\Big(\hat{\Omega}(u) + \sigma_S \sigma_\mu \rho\Big)}{\sigma_S} d\hat{B}_u.$$

similar to the proofs of Corollary 3.5.14 and Corollary 3.5.16 in *Karatzas and Shreve* [41], Beneš' condition implies $\widetilde{H}$ is a true martingale with respect to $(\Omega, \mathcal{F}^S, \mathbb{P})$,

Now define the probability measure $\widetilde{\mathbb{P}}$ as

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \widetilde{H}_T,$$

Girsanov theorem states that

$$\widetilde{B}_t \triangleq \hat{B}_t + \int_0^t \frac{\hat{\mu}_v}{\sigma_S} dv, \quad 0 \leq t \leq T$$

is a Brownian Motion under $(\widetilde{\mathbb{P}}, (\mathcal{F}_t^S)_{0 \leq t \leq T})$.

Then we can rewrite the wealth process dynamics as:

$$X_T + \int_0^T c_v dv = x + \int_0^T \pi_v \sigma_S d\widetilde{B}_v,$$

Since we have $X_T \geq 0$, it's easy to see that $\int_0^t \pi_v \sigma_S d\widetilde{W}_v$ is a supermartingale under $(\Omega, \mathbb{F}^S, \widetilde{\mathbb{P}})$, and take the expectation under $\widetilde{\mathbb{P}}$, we have:

$$x_0 \geq \widetilde{\mathbb{E}}\Big[\int_0^T c_v dv\Big].$$

Follow the inequality (3.4.5), we will further have:

$$x_0 \geq z_0 \widetilde{\mathbb{E}}\Big[\int_0^T \exp\Big(\int_0^v (\delta(u) - \alpha(u)) du\Big) dv\Big].$$

Since $\delta(t)$ and $\alpha(t)$ are deterministic functions, we easily arrive $x_0 \geq W(0) z_0$.

In general, for $\forall t \in [0, T]$, follow the same procedure, we can then take conditional expectation under filtration $\mathcal{F}_t^S$, and get

$$X_t \geq Z_t \widetilde{\mathbb{E}}\Big[\int_t^T \exp\Big(\int_t^v (\delta(u) - \alpha(u)) du\Big) dv \Big| \mathcal{F}_t^S\Big],$$

again since $\delta(t), \alpha(t)$ are deterministic, we obtain $X_t \geq W(t)Z_t, \quad 0 \leq t \leq T$. $\square$

Remark 3.4.1. *The constraint on the controlled wealth process X_t and the habit formation process Z_t agrees with the effective domain $\{(x,z) \in (0,\infty) \times [0,\infty) : x \geq W(t)z\}$ of the HJB equation for the values of x and z. Aside from the consequence that the process $V(t, X_t, Z_t, \hat{\mu}_t)$ is therefore well defined, it plays a critical role in our following proof of the verification lemma.*

Remark 3.4.2. *We now revisit the quantity $\mathbb{E}\Big[\int_0^T w_t Y_t dt\Big]$ where we define $w_t \triangleq e^{\int_0^t (\delta_v - \alpha_v)dv}$ by (2.3.15) and process $Y_t \in \mathcal{M}$ is the equivalent local martingale measure density process of the financial market under $(\Omega, \mathbb{F}^S, \mathbb{P})$. The process $\widetilde{H}_t$ defined by (3.4.6) guarantees the set $\mathcal{M}$ is not empty. As processes δ_t and α_t are now both deterministic functions, it implies that*

$$\mathbb{E}\Big[\int_0^T w_t Y_t dt\Big] = \int_0^T e^{\int_0^t (\delta_v - \alpha_v)dv} dt = W(0),$$

where we define the deterministic function $W(t)$ by (3.3.16), moreover, we actually have

$$W(t) = \frac{1}{w_t Y_t}\mathbb{E}\Big[\int_t^T w_s Y_s ds \Big| \mathcal{F}_t^S\Big] = \int_t^T \frac{w_s}{w_t} ds. \tag{3.4.7}$$

Under the partial observation filtration, by applying the consumption Budget Constraint in Chapter 2, we obtain the inequality $x_0 \geq z_0 \mathbb{E}\Big[\int_0^T w_t Y_t dt\Big]$ for all $Y_t \in \mathcal{M}$, which is equivalent to our condition $x_0 \geq z_0 W(0)$. On the other hand, we can also see why the deterministic function $W(t)$ in our HJB equation is independent of the variable η and parameter θ_0 by reading that *the right-hand side of the equation* (3.4.7) *does not dependent on the choice of equivalent local martingale measure density process* Y_t*, and hence* $W(t)$ *is evidently independent of the value of the estimated conditional drift process* $\hat{\mu}_t$ *and conditional variance* $\hat{\Omega}(t;\theta_0)$*.*

3.4.1 The Case $p < 0$

(THE PROOF OF THEOREM 3.3.3*).*

First, for any pair of admissible control $(\pi_t, c_t) \in \mathcal{A}$, *Ito's* lemma gives

$$d\Big[V(t, X_t, Z_t, \hat{\mu}_t)\Big] = \Big[\mathcal{G}^{\pi_t, c_t} V(t, X_t, Z_t, \hat{\mu}_t)\Big]dt + \Big[V_x \sigma_S \pi_t + V_\eta \frac{\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)}{\sigma_S}\Big]d\hat{B}_t, \tag{3.4.8}$$

where we define the process $\mathcal{G}^{\pi_t, c_t} V(t, X_t, Z_t, \hat{\mu}_t)$ as

$$\begin{aligned}\mathcal{G}^{\pi_t, c_t} V(t, X_t, Z_t, \hat{\mu}_t) &= V_t - \alpha(t) Z_t V_z - \lambda(\hat{\mu}_t - \bar{\mu}) V_\eta + \frac{\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)^2}{2\sigma_S^2} V_{\eta\eta} \\ &- c_t V_x + c_t \delta(t) V_z + \frac{(c_t - Z_t)^p}{p} + \pi_t \hat{\mu}_t V_x + \frac{1}{2}\sigma_S^2 \pi_t^2 V_{xx} + V_{x\eta}\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)\pi_t.\end{aligned}$$

Recall $V(t, x, z, \eta)$ is the classical solution of HJB equation (3.3.12), choose the localizing sequence τ_n, we integrate the equation (3.4.8) on $[0, \tau_n \wedge T]$, and take the expectation under probability measure $\mathbb{P}_{x_0, z_0, \eta_0, \theta_0}$, and let's denote $\mathbb{E} = \mathbb{E}_{x_0, z_0, \eta_0, \theta_0}$, we have

$$V(0, x_0, z_0, \eta_0) \geq \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds\Big] + \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T})\Big]. \tag{3.4.9}$$

Now, we follow the idea by *Janeček and Sîrbu* [36], let's fix this pair of control choice $(\pi_t, c_t) \in \mathcal{A} = \mathcal{A}_{x_0}$, where we denote $\mathcal{A}_{x_0}$ as the admissible space with initial endowment x_0. And for $\forall\epsilon > 0$, it is clear that $\mathcal{A}_{x_0} \subseteq \mathcal{A}_{x_0+\epsilon}$, and $(\pi_t, c_t) \in \mathcal{A}_{x_0+\epsilon}$. Also it is clear that $X_t^{x_0+\epsilon} = X_t^{x_0} + \epsilon = X_t + \epsilon, \quad 0 \le t \le T$. Follow the same procedure above, and notice process Z_t keeps the same under the consumption policy c_t, then under probability measure $\mathbb{P}_{x_0,z_0,\eta_0}$, we can obtain:

$$\begin{aligned} V(0, x_0 + \epsilon, z_0, \eta_0) \ge \mathbb{E}\Big[&\int_0^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds\Big] \\ &+ \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n\wedge T} + \epsilon, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big]. \end{aligned} \tag{3.4.10}$$

By Monotone Convergence Theorem, we first know:

$$\lim_{n\to+\infty} \mathbb{E}\Big[\int_0^{\tau_n\wedge T} \frac{(c_s - Z_s)^p}{p} ds\Big] = \mathbb{E}\Big[\int_0^{T} \frac{(c_s - Z_s)^p}{p} ds\Big]. \tag{3.4.11}$$

For simplicity, let's denote $Y_t = \Big(X_t - W(t)Z_t\Big)$, we know by definition (3.3.34) that:

$$V(\tau_n \wedge T, X_{\tau_n\wedge T} + \epsilon, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T}) = \frac{1}{p}(Y_{\tau_n\wedge T} + \epsilon)^p N_{\tau_n\wedge T}^{1-p}.$$

Proposition 3.4.1 gives $X_t \ge W(t)Z_t$ for $0 \le t \le T$ under any admissible control pair (π_t, c_t), we know $Y_{\tau_n\wedge T} + \epsilon \ge \epsilon > 0, \quad \forall 0 \le t \le T$. Since also $p < 0$, we will have

$$\sup_n (Y_{\tau_n\wedge T} + \epsilon)^p < \epsilon^p < +\infty. \tag{3.4.12}$$

Now from Remark $A.1.1$, we already derived that $A(t;s) \le 0, \ \forall 0 \le t \le s \le T$. Combining this with the fact that $W(s)$, $\delta(s)$ are continuous and hence bounded on $[0, T]$ and when $p < 0$, we also have $1 - a(t;s)\hat{\Omega}(t) > 0$ and $1 - f(t;s)\hat{\Omega}(t) > 0$ as well as $a(t;s)$, $b(t;s)$, $c(t;s)$, $f(t;s)$ and $g(t;s)$ are all bounded for $0 \leq t \leq s \leq T$, we deduce that the explicit solutions $B(t;s)$ and $C(t;s)$ are both bounded on $0 \leq t \leq s \leq T$, hence we have:

$$N(0,\eta) \leq k_1 \exp(k\eta) \;\text{ for some large constants }\; k, k_1 > 1,$$

which shows the existence of some constants $\bar{k}, \bar{k}_1 > 1$ such that

$$\sup_n N_{\tau_n \wedge T}^{1-p} \leq \sup_{t\in[0,T]} \Big(k_1 \exp\Big(k\hat{\mu}_t\Big)\Big)^{1-p} \leq \bar{k}_1 \exp\Big(\bar{k} \sup_{t\in[0,T]} \hat{\mu}_t\Big).$$

We recall that $\hat{\mu}_t$ satisfies the Ornstein Uhlenbeck diffusion (3.3.2), which gives:

$$\hat{\mu}_t = e^{-t\lambda}\eta + \bar{\mu}(1-e^{-t\lambda}) + \int_0^t e^{\lambda(u-t)} \frac{\Big(\hat{\Omega}(u) + \sigma_S\sigma_\mu\rho\Big)}{\sigma_S} d\hat{B}_u.$$

Hence, there exists positive constants l and $l_1 > 1$ large enough, such that:

$$\sup_{t\in[0,T]} \hat{\mu}_t \leq l + \sup_{t\in[0,T]} l_1\hat{B}_t, \quad t \in [0,T].$$

Using the distribution of running maximum of the Brownian Motion, there exists some positive constants $\bar{l} > 1$ and $\bar{l}_1$ such that

$$\mathbb{E}\Big[\sup_n N_{\tau_n\wedge T}^{1-p}\Big] \leq \bar{l}_1 \mathbb{E}\Big[\exp\big(\sup_{t\in[0,T]} \bar{l}\hat{B}_t\big)\Big] < +\infty. \tag{3.4.13}$$

At last, by the above (3.4.12) and (3.4.13), we can conclude that

$$\mathbb{E}\Big[\sup_n V(\tau_n \wedge T, X_{\tau_n\wedge T} + \epsilon, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big] < +\infty.$$

By virtue of Dominated Convergence Theorem, we can deduce:

$$\begin{aligned} \lim_{n\to+\infty} \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n\wedge T} + \epsilon, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big] &= \mathbb{E}\Big[\frac{1}{p}(Y_T+\epsilon)^p N(T,\hat{\mu}_T)\Big] \\ &= \mathbb{E}\Big[\frac{(X_T+\epsilon)^p}{p}\Big] > \mathbb{E}\Big[\frac{X_T^p}{p}\Big]. \end{aligned}$$

Combine this with equation (3.4.10), and notice the pair of control $(\pi_t, c_t) \in \mathcal{A}$, we will see that:

$$V(0, x_0 + \epsilon, z_0, \eta_0; \theta_0) \geq \sup_{\pi, c \in \mathcal{A}} \mathbb{E}\Big[\int_0^T \frac{(c_s - Z_s)^p}{p} ds + \frac{X_T^p}{p}\Big] = v(x_0, z_0, \eta_0, \theta_0).$$

Notice $V(t, x, z, \eta; \theta_0)$ is continuous in variable x, and since $\epsilon > 0$ is arbitrary, we can take the limit as:

$$V(0, x_0, z_0, \eta_0; \theta_0) = \lim_{\epsilon \to 0} V(0, x_0 + \epsilon, z_0, \eta_0) \geq v(x_0, z_0, \eta_0, \theta_0).$$

On the other hand, for π_t^* and c_t^* defined by (3.3.38) and (3.3.39) respectively, we first want to show the SDE for wealth process:

$$dX_t^* = (\pi_t^* \mu_t - c_t^*)dt + \sigma_S \pi_t^* d\hat{B}_t, \quad 0 \leq t \leq T, \tag{3.4.14}$$

with initial condition $x_0 > W(0)z_0$ has a unique strong solution and also satisfies $X_t^* > W(t)Z_t^*, \ \forall t \in [0, T]$.

Denote $Y_t^* = X_t^* - W(t)Z_t^*$, and apply *Ito's* lemma and substitute π_t^* as defined by (3.3.38), we can get:

$$\begin{aligned} dY_t^* =& \Big[\pi_t^* \hat{\mu}_t - c_t^* - W_t(t)Z_t^* - W(t)\delta(t)c_t^* + W(t)\alpha(t)Z_t^*\Big]dt + \pi_t^* \sigma_S d\hat{B}_t \\ =& \Big[\Big(- W_t(t) + W(t)\alpha(t)\Big)Z_t^* - (1 + W(t)\delta(t))c_t^* + \frac{\hat{\mu}_t^2}{(1-p)\sigma_S^2} Y_t^* \\ &+ \frac{\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)}{\sigma_S^2} \frac{N_\eta}{N} \hat{\mu}_t Y_t^*\Big] dt + \Big[\frac{\hat{\mu}_t}{(1-p)\sigma_S} + \frac{\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)}{\sigma_S} \frac{N_\eta}{N}\Big] Y_t^* d\hat{B}_t. \end{aligned} \tag{3.4.15}$$

Recall the definition of $W(t)$ by (3.3.16) and substitute c_t^* defined by (3.3.39) into (3.4.15) above, we will further have

$$\begin{aligned} dY_t^* =& \Big[-\frac{\Big(1+\delta(t)W(t)\Big)^{\frac{-p}{1-p}}}{N} + \frac{\hat{\mu}_t^2}{(1-p)\sigma_S^2} + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)}{\sigma_S^2}\frac{N_\eta}{N}\hat{\mu}_t\Big] Y_t^* dt \\ &+ \Big[\frac{\hat{\mu}_t}{(1-p)\sigma_S} + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)}{\sigma_S}\frac{N_\eta}{N}\Big] Y_t^* d\hat{B}_t. \end{aligned}$$

In order to solve X_t^* in a more explicit formula, we define the auxiliary process by:

$$\Gamma_t = \frac{N(t,\hat{\mu}_t)}{Y_t^*}, \quad \forall 0 \le t \le T.$$

By Itô's lemma, we can derive the SDE for process Γ_t as:

$$\begin{aligned} d\Gamma_t =& \frac{\Gamma_t}{N}\Big[N_t - \lambda(\hat{\mu}_t - \bar{\mu})N_\eta + \frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2} N_{\eta\eta} + \frac{\hat{\mu}_t\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)p}{(1-p)\sigma_S^2} N_\eta \\ &+ \Big(1+\delta(t)W(t)\Big)^{\frac{-p}{1-p}} + \frac{p\hat{\mu}_t^2}{(1-p)^2\sigma_S^2} N\Big] dt + \Gamma_t\Big[\frac{-\hat{\mu}_t}{(1-p)\sigma_S}\Big] d\hat{B}_t. \end{aligned} \tag{3.4.16}$$

Recall that $N(t,\eta)$ satisfies the linear PDE (3.3.21), we can simplify (3.4.16) to be:

$$d\Gamma_t = \Gamma_t\Big[\frac{p\hat{\mu}_t^2}{2(1-p)^2\sigma_S^2}\Big] dt + \Gamma_t\Big[\frac{-\hat{\mu}_t}{(1-p)\sigma_S}\Big] d\hat{B}_t.$$

Hence, we can finally get the above SDE has a unique strong solution as:

$$\Gamma_t = \Gamma_0 \exp\Big(-\int_0^t \frac{\hat{\mu}_u^2}{2(1-p)\sigma_S^2} du - \int_0^t \frac{\hat{\mu}_u}{(1-p)\sigma_S} d\hat{B}_u \Big),$$

Initial condition $\Gamma_0 = \frac{N(0,\eta)}{x_0 - W(0)z_0} > 0$ implies $\Gamma_t > 0, \ \forall\, 0 \le t \le T$. And, hence, we finally proved that the SDE (3.4.14) has a unique strong solution defined by (3.3.40) and the solution X_t^* satisfies the wealth process constraint (3.4.1)

Now, we proceed to verify π_t^* and c_t^* are actually in the admissible space $\mathcal{A}$.

First, by the definition (3.3.38) and (3.3.39), it's clear that π_t^* and c_t^* are $\mathcal{F}_t^S$ progressively measurable, and by the path continuity of $Y_t^* = X_t^* - W(t)Z_t^*$, hence, of π_t^* and c_t^*, it's easy to show that:

$$\int_0^T (\pi_t^*)^2 dt < +\infty, \quad and \int_0^T c_t^* dt < +\infty, \quad a.s.$$

Also, since $X_t^* > W(t)Z_t^*,\ \forall t \in [0,T]$, by the definition of c_t^*, we know the consumption constraint $c_t^* > Z_t^*, \quad \forall t \in [0,T]$ is satisfied. And hence $(\pi_t^*, c_t^*) \in \mathcal{A}$.

Given the pair of control policy (π_t^*, c_t^*) as above, following the same steps and the definition of stopping time τ_n, instead of (3.4.9), we can now instead get the equality:

$$V(0, x_0, z_0, \eta_0; \theta_0) = \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_t^* - Z_t^*)^p}{p} dt\Big] + \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n \wedge T}^*, Z_{\tau_n \wedge T}^*, \hat{\mu}_{\tau_n \wedge T})\Big].$$

And hence, we apply Monotone Convergence Theorem again:

$$\lim_{n \to +\infty} \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_t^* - Z_t^*)^p}{p} dt\Big] = \mathbb{E}\Big[\int_0^T \frac{(c_t^* - Z_t^*)^p}{p} dt\Big],$$

When $p < 0$, we have function $V(t, x, z, \eta) < 0$ by it's definition, and by Fatou's lemma,

$$\lim_{n \to +\infty} \sup \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n \wedge T}^*, Z_{\tau_n \wedge T}^*, \hat{\mu}_{\tau_n \wedge T})\Big] \le \mathbb{E}\Big[V(T, X_T^*, Z_T^*, \hat{\mu}_T)\Big] = \mathbb{E}\Big[\frac{(X_T^*)^p}{p}\Big].$$

Therefore, it gives

$$V(0,x_0,z_0,\eta_0;\theta_0) \le \mathbb{E}\Big[\int_0^T \frac{(c_t^* - Z_t^*)^p}{p}dt + \frac{(X_T^*)^p}{p}\Big] \le v(x_0,z_0,\eta_0,\theta_0)$$

which completes the proof. □

3.4.2 The Case: *0 < p < 1*

We proceed to prove the following two Lemmas which play important roles in the proof of the second part of our main result.

Lemma 3.4.2. *If constant $k > 0$ satisfies:*

$$k < \frac{\lambda^2\sigma_S^2}{2(\Theta + \sigma_S\sigma_\mu\rho)^2} \tag{3.4.17}$$

for any $t \ge 0$, there exists a constant Λ_1 such that

$$\mathbb{E}\Big[\exp\Big(\int_0^t k\hat{\mu}_s^2 ds\Big)\Big] \le \Lambda_1 < +\infty.$$

Proof. It is easy to choose an increasing sequence of smooth functions $Q_n(y) \nearrow ky^2$ as $n \to \infty$ such that $0 \le Q_n(y) \le n$ with $\Big|Q_n'(y)\Big|$ and $\Big|Q_n''(y)\Big|$ uniformly bounded. And for each fixed $t \ge 0$ and η, we define:

$$\phi(t,\eta) = \mathbb{E}\Big[\exp\Big(\int_0^t Q_n(\hat{\mu}_s)ds\Big)\Big],$$

where $\hat{\mu}_0 = \eta$.

Similar to the proof of Feynman-Kac formula, the function $\phi(t,\eta)$ is a classical solution of the linear parabolic equation:

$$\phi_t = \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}\phi_{\eta\eta} - \lambda(\eta - \bar{\mu})\phi_\eta + Q_n(\eta)\phi, \tag{3.4.18}$$

with initial condition $\phi(0, \eta) = 1$. See also Lemma 1.12 in *Pang* [59] for details.

First, it's clear that constant 0 is a subsolution of the above equation. Moreover, under assumption (3.4.17), it's easy to show that for each fixed $t \geq 0$, the equation:

$$\frac{2\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{\sigma_S^2}x^2 - 2\lambda x + k = 0$$

has two positive real roots

$$x_1 = \frac{\lambda - \sqrt{\lambda^2 - \frac{2(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}k}}{2\frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}}, \quad \text{and} \quad x_2 = \frac{\lambda + \sqrt{\lambda^2 - \frac{2(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}k}}{2\frac{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}}.$$

And for any positive constant a such that:

$$0 < a < \frac{\lambda + \sqrt{\lambda^2 - \frac{(\Theta_2+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}k}}{2\frac{(\Theta_1+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2}},$$

with $\Theta_1 = \max(\theta, \theta^*)$ and $\Theta_2 = \min(\theta, \theta^*)$, and the positive constant b such that:

$$b > a\frac{(\Theta_1 + \sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} - \frac{\lambda^2\bar{\mu}^2a^2}{2a^2\frac{(\Theta_1+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} - 2a\lambda + k}.$$

It's easy to verify that $f(t, \eta) = \exp(bt + a\eta^2)$ satisfies:

$$f_t \geq \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}f_{\eta\eta} - \lambda(\eta - \bar{\mu})f_\eta + k\eta^2,$$

with the initial condition $f(0, \eta) \geq 1$.

And since $Q_n(\eta) < k\eta^2$, we get function $f(t, \eta)$ is the supersolution of the equation (3.4.18), and $\Big\langle 0, f(t, \eta)\Big\rangle$ is the coupled subsolution and supersolution. Theorem 7.2 from *Pao* [60] shows that function $\phi(t, \eta)$ satisfies:

$0 \le \phi(t,\eta) \le f(t,\eta) \equiv \Lambda_1$, and hence Monotone Convergence Theorem leads to:

$$\mathbb{E}\Big[\exp\Big(\int_0^t k\hat{\mu}_s^2 ds\Big)\Big] \le \Lambda_1 < +\infty.$$

□

Lemma 3.4.3. *If constant $\bar{k} > 0$ satisfies*

$$\bar{k} < \frac{\lambda\sigma_S^2}{(\Theta + \sigma_S\sigma_\mu\rho)^2}, \tag{3.4.19}$$

for fixed constant $\kappa > 0$, there exists a constant Λ_2 independent of t, and

$$\mathbb{E}\Big[\exp\Big(\bar{k}(\hat{\mu}_t + \kappa)^2\Big)\Big] \le \Lambda_2 < \infty, \quad t \in [0,T].$$

Proof. Similar to the proof of Lemma 3.4.2, we again construct an increasing sequence of functions $\{Q_n(y)\}$ for $n \in \mathbb{N}$ such that $\lim_{n\to+\infty} Q_n(y) = \bar{k}(y+\kappa)^2$. And for each fixed $t \in [0,T]$ and η, we define:

$$\psi(t,\eta) = \mathbb{E}\Big[\exp\Big(Q_n(\hat{\mu}_t)\Big)\Big],$$

where $\hat{\mu}_0 = \eta$.

Then a direct corollary of Theorem 5.6.1 of *Friedman* [29] gives the function $\psi(t,\eta)$ is a classical solution of the linear parabolic equation:

$$\psi_t = \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}\psi_{\eta\eta} - \lambda(\eta - \bar{\mu})\psi_\eta, \tag{3.4.20}$$

with initial condition $\psi(0,\eta) = e^{Q_n(\eta)}$.

Under assumption (3.4.19), and choose any constant a such that

$$\bar{k} < a < \frac{\lambda\sigma_S^2}{(\Theta + \sigma_S\sigma_\mu\rho)^2},$$

where $x_1 = \frac{\lambda\sigma_S^2}{(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho)^2}$ is one real root of the algebraic equation:

$$\frac{2\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{\sigma_S^2}x^2 - 2\lambda x = 0$$

for each fixed $t \in [0, T]$. And choose any positive constant b such that

$$\begin{aligned} b >& a\frac{(\Theta + \sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} + \frac{2a^2(\Theta + \sigma_S\sigma_\mu\rho)^2\kappa^2}{\sigma_S^2} + 2a\lambda\bar{\mu}\kappa \\ &- \frac{\Big(\frac{2a^2\kappa(\Theta+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} - a\lambda\kappa - a\lambda\bar{\mu}\Big)^2}{2a^2\frac{(\Theta+\sigma_S\sigma_\mu\rho)^2}{\sigma_S^2} - 2a\lambda}, \end{aligned}$$

It is easy to verify that $f(t, \eta) = \exp(bt + a(\eta + \kappa)^2)$ satisfies

$$f_t \geq \frac{\Big(\hat{\Omega}(t) + \sigma_S\sigma_\mu\rho\Big)^2}{2\sigma_S^2}f_{\eta\eta} - \lambda(\eta - \bar{\mu})f_\eta,$$

with the initial condition $f(0, \eta) = e^{a(\eta+\kappa)^2} \geq \psi(0, \eta)$, hence we get the function $f(t, \eta)$ is the supersolution of the equation (3.4.20), and it is trivial to show $g(t, \eta) \equiv 0$ is the subsolution, therefore $\big\langle 0, f(t, \eta)\big\rangle$ are the coupled subsolution and supersolution. Again by Theorem 7.2 from *Pao* [60], that function $\psi(t, \eta)$ satisfies: $0 \leq \psi(t, \eta) \leq f(t, \eta) \leq e^{bT+a(\eta+\kappa)^2} \equiv \Lambda_2$, hence Monotone Convergence Theorem implies:

$$\mathbb{E}\Big[\exp\Big(\bar{k}(\hat{\mu}_t + \kappa)^2\Big)\Big] \leq \Lambda_2 < +\infty, \quad \forall t \in [0, T].$$

□

(THE PROOF OF THEOREM 3.3.3*, CONTINUED).*

For any pair of admissible control $(\pi_t, c_t) \in \mathcal{A}$, similar to the case for

$p < 0$, choose the same localizing sequence τ_n such that

$$V(0, x_0, z_0, \eta_0) \geq \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds\Big] + \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T})\Big]. \tag{3.4.21}$$

Now, by monotone convergence theorem, we first know:

$$\lim_{n \to +\infty} \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_s - Z_s)^p}{p} ds\Big] = \mathbb{E}\Big[\int_0^{T} \frac{(c_s - Z_s)^p}{p} ds\Big].$$

And for $0 < p < 1$, $V(t, x, z, \eta) \geq 0$ for all $t \in [0, T]$ by the definition (3.3.34) and (3.4.1), and Fatou's lemma yields that:

$$\lim_{n \to +\infty} \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n \wedge T}, Z_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T})\Big] \geq \mathbb{E}\Big[V(T, X_T, Z_T, \hat{\mu}_T)\Big] = \mathbb{E}\Big[\frac{X_T^p}{p}\Big],$$

which implies that:

$$V(0, x_0, z_0, \eta_0) \geq \sup_{\pi, c \in \mathcal{A}} \mathbb{E}\Big[\int_0^{T} \frac{(c_s - Z_s)^p}{p} ds + \frac{X_T^p}{p}\Big] = v(x_0, z_0, \eta_0, \theta_0).$$

On the other hand, for the π_t^* and c_t^* defined by (3.3.38), (3.3.39), again follow the same procedure in the proof for case $p < 0$, we can show π_t^* and c_t^* are actually in the admissible space $\mathcal{A}$.

Now, by policies π_t^* and c_t^*, similarly, we can now get the equality:

$$V(0, x_0, z_0, \eta_0) = \mathbb{E}\Big[\int_0^{\tau_n \wedge T} \frac{(c_s^* - Z_s^*)^p}{p} ds\Big] + \mathbb{E}\Big[V(\tau_n \wedge T, X^*_{\tau_n \wedge T}, Z^*_{\tau_n \wedge T}, \hat{\mu}_{\tau_n \wedge T})\Big].$$

By the definition of $V(t, x, z, \eta)$, we know that:

$$V(T \wedge \tau_n, X^*_{T \wedge \tau_n}, Z^*_{T \wedge \tau_n}, \hat{\mu}_{T \wedge \tau_n}) \leq k_1\Big[\Big(\frac{Y^*}{N}\Big)^{2p}_{T \wedge \tau_n} + N^2(T \wedge \tau_n, \hat{\mu}_{T \wedge \tau_n})\Big]$$

for some positive constants k_1, which are independent of n.

For the first term, we notice that

$$\begin{aligned}\Big(\frac{Y^*}{N}\Big)^{2p}_{T\wedge\tau_n} \leq & \frac{1}{2}\Big[\exp\Big(\int_0^{T\wedge\tau_n}\frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{B}_u - \int_0^{T\wedge\tau_n}\frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big) \\ & + \exp\Big(\int_0^{T\wedge\tau_n}\frac{2(p^2+p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]\end{aligned}$$

and hence, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_n\Big(\frac{Y^*}{N}\Big)^{2p}_{T\wedge\tau_n}\Big] \leq & \frac{1}{2}\mathbb{E}\Big[\sup_n\exp\Big(\int_0^{T\wedge\tau_n}\frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{B}_u - \int_0^{T\wedge\tau_n}\frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big) \\ & + \sup_n\exp\Big(\int_0^{T\wedge\tau_n}\frac{2(p^2+p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]\end{aligned}$$

and again since $\hat{\mu}_t$ follows the dynamics (3.3.2), by Beneš' condition (see Corollary 3.5.14 and Corollary 3.5.16 in *Karatzas and Shreve* [41]), we see that the exponential local martingale $M_t = \exp\Big(\int_0^t \frac{2p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{B}_u - \int_0^t \frac{2p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)$ is a true martingale, and hence, by Doob's maximal inequality, we first derive that

$$\begin{aligned}&\mathbb{E}\Big[\sup_n\exp\Big(\int_0^{T\wedge\tau_n}\frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{B}_u - \int_0^{T\wedge\tau_n}\frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big] \\ \leq & \mathbb{E}\Big[\sup_{t\in[0,T]}\exp\Big(\int_0^{t}\frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{B}_u - \int_0^{t}\frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big] \\ \leq & k(p)\mathbb{E}\Big[\exp\Big(\int_0^{T}\frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{B}_u - \int_0^{T}\frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]\end{aligned}$$

where $k(p)$ is a constant depending on p. Moreover, similar to the proofs of Corollary 3.5.14 and Corollary 3.5.16 in *Karatzas and Shreve* [41], Corollary 1 and Corollary 2 in *Grigelionis and Mackevičius* [31] further states that the true martingale M_t defined as above satisfies the finite moments property, i.e., for any $r > 1$, we have $\mathbb{E}\Big[M_T^r\Big] < \infty$. Hence we can conclude that for $r = 2$,

$$\mathbb{E}\Big[\exp\Big(\int_0^{T}\frac{4p\hat{\mu}_u}{(1-p)\sigma_S}d\hat{B}_u - \int_0^{T}\frac{4p^2\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big] < \infty.$$

For the second part, we can therefore apply Assumption (3.3.35) and Lemma 3.4.2, and it yields that:

$$\begin{aligned}\mathbb{E}\Big[\sup_n \exp\Big(\int_0^{T\wedge\tau_n} \frac{(2p^2+2p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big] &\le \mathbb{E}\Big[\exp\Big(\int_0^{T} \frac{(2p^2+2p)\hat{\mu}_u^2}{(1-p)^2\sigma_S^2}du\Big)\Big]\\ &< \Lambda_1 < +\infty,\end{aligned}$$

for some constant $\Lambda_1 > 0$.

We now recall that under Assumption (3.3.31), Lemma 3.3.2 implies that there exists constants k, k_1 such that

$$N(t,\eta) \le k e^{\bar{K}_1(\eta+k_1)^2},$$

where $A(t;s) \le \bar{K}_1$ for all $0 \le t \le s \le T$, and hence we have

$$\sup_n\Big(N^2(T\wedge\tau_n, \hat{\mu}_{T\wedge\tau_n})\Big) \le \sup_{t\in[0,T]} k e^{2\bar{K}_1(\hat{\mu}_t+k_1)^2}.$$

Then we just need to show that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} k e^{2\bar{K}_1(\hat{\mu}_t+k_1)^2}\Big] < \infty. \tag{3.4.22}$$

Define $\varphi(x) \triangleq e^{2\bar{K}_1(x+k_1)^2}$ and apply *Ito's* lemma, we have

$$\begin{aligned}d\varphi(\hat{\mu}_t) =&\varphi(\hat{\mu}_t)\Big[\Big(-4\bar{K}_1\lambda + 8\bar{K}_1^2\frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{\sigma_S^2}\Big)\hat{\mu}_t^2 + 4\bar{K}_1\lambda\bar{\mu}\hat{\mu}_t\\ &+ 2\bar{K}_1\frac{\Big(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\Big)^2}{\sigma_S^2}\Big]dt + dL_t.\end{aligned}$$

Assumption (3.3.36) guarantees $-4\bar{K}_1\lambda + 8\bar{K}_1^2\frac{\left(\hat{\Omega}(t)+\sigma_S\sigma_\mu\rho\right)^2}{\sigma_S^2} < 0$, and hence there exists an upper bound constant $k_2 > 0$ such that

$$d\varphi(\hat{\mu}_t) \le \varphi(\hat{\mu}_t)k_2 dt + dL_t,$$

where the local martingale part is:

$$dL_t \triangleq \varphi(\hat{\mu}_t) 4\bar{K}_1 \hat{\mu}_t \frac{\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)}{\sigma_S} d\hat{B}_t.$$

From which we can derive that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \varphi(\hat{\mu}_t)\Big] \leq \varphi(\eta) + \int_0^T k_2 \mathbb{E}\Big[\sup_{s\in[0,t]} \varphi(\hat{\mu}_s)\Big] dt + \mathbb{E}\Big[\sup_{t\in[0,T]} L_t\Big],$$

Burhholder-Davis-Gundy Inequality and Jensen's Inequality induce that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} L_t\Big] \leq k_3 \Big(\int_0^T \mathbb{E}\Big[16\bar{K}_1^2 \frac{\Big(\hat{\Omega}(t) + \sigma_S \sigma_\mu \rho\Big)^2}{\sigma_S^2} \hat{\mu}_t^2 e^{4\bar{K}_1(\hat{\mu}_t + k_1)^2}\Big] dt\Big)^{\frac{1}{2}},$$

However, under Assumption (3.3.36), there exists a constant $\epsilon > 0$ such that $4\bar{K}_1 + \epsilon < \frac{\lambda \sigma_S^2}{(\Theta + \sigma_S \sigma_\mu \rho)^2}$, and by Hölder's Inequality, choose the conjugates $q = \frac{4\bar{K}_1 + \epsilon}{4\bar{K}_1}$ and $\frac{1}{q} + \frac{1}{q'} = 1$, then

$$\mathbb{E}\Big[\hat{\mu}_t^2 e^{4\bar{K}_1(\hat{\mu}_t + k_1)^2}\Big] \leq \Big(\mathbb{E}\Big[\hat{\mu}_t^{2q'}\Big]\Big)^{\frac{1}{q'}} \Big(\mathbb{E}\Big[e^{(4\bar{K}_1 + \epsilon)(\hat{\mu}_t + k_1)^2}\Big]\Big)^{\frac{1}{q}},$$

and by Lemma 3.4.3, there exists a constant Λ_2 independent of t such that

$$\mathbb{E}\Big[e^{(4\bar{K}_1 + \epsilon)(\hat{\mu}_t + k_1)^2}\Big] \leq \Lambda_2 < \infty, \ \forall t \in [0, T].$$

And again by the fact that there exists positive constants l and $l_1 > 1$ large enough, such that:

$$\sup_{t\in[0,T]} \hat{\mu}_t \leq l + \sup_{t\in[0,T]} l_1 \hat{B}_t, \quad t \in [0, T],$$

we also obtain

$$\int_0^T \Big(\mathbb{E}\Big[\hat{\mu}_t^{2q'}\Big]\Big)^{\frac{1}{q'}} dt \leq T\Big(\mathbb{E}\Big[(l + \sup_{t\in[0,T]} l_1 \hat{B}_t)^{2q'}\Big]\Big)^{\frac{1}{q'}} < \infty,$$

due to the distribution of running maximum of the Brownian motion $\hat{W}_t$. Hence we get the boundedness of $\mathbb{E}\Big[\sup_{t\in[0,T]} L_t\Big] \le k_4 < \infty$ for some constant k_4, and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \varphi(\hat{\mu}_t)\Big] \le \varphi(\eta) + \int_0^T k_2 \mathbb{E}\Big[\sup_{s\in[0,t]} \varphi(\hat{\mu}_s)\Big] dt + k_4,$$

The Gronwall's Inequality verifies (3.4.22).

Therefore, putting all pieces together, we eventually derived that

$$\mathbb{E}\Big[\sup_n V(\tau_n \wedge T, X_{\tau_n\wedge T}, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big] < \infty \tag{3.4.23}$$

and Dominated Convergence Theorem leads to:

$$\lim_{n\to\infty} \mathbb{E}\Big[V(\tau_n \wedge T, X_{\tau_n\wedge T}, Z_{\tau_n\wedge T}, \hat{\mu}_{\tau_n\wedge T})\Big] = \mathbb{E}\Big[\frac{(X_T^*)^p}{p}\Big].$$

Together with Monotone Convergence Theorem, we deduce

$$V(0, x_0, z_0, \eta_0; \theta_0) = \mathbb{E}\Big[\int_0^T \frac{(c_s^* - Z_s^*)^p}{p} ds + \frac{(X_T^*)^p}{p}\Big] \le v(x_0, z_0, \eta_0, \theta_0),$$

which completes the proof. □

Chapter 4

Future Research

In this section, we plan to shed some light on the possible extensions of our current work in various directions. As we pointed out in the Chapter of Introduction of this dissertation, we are only considering a special family of consumption habit formation preference, namely, the addictive linear habit formation. A natural open problem arise when we abandon the linearity structure and habit addiction constraint. It is not surprising that the problem of existence of optimal consumption policy under general habit formation preference does not fit into our previous framework, and has to be investigated as a separate project. We mainly discuss three interesting future research plans as follows:

1. The Case of Nonaddictive Linear Habit Formation

As the pioneering work, *Detemple and Karatzas* [23] studied utility maximization problem with the non-addictive linear habit formation preference $\mathbb{E}[\int_0^T U(t, c_t - Z_t)dt]$ in the complete market driven by Itô processes, where instead they define $U : [0, \infty) \times (-\infty, \infty) \to \mathbb{R}$, for example when U is an exponential utility function. Their consumption c_t is required to be non-negative but is allowed to fall below the *"the standard of living"* index

Z_t that aggregates past consumption. They provided a constructive proof for the existence of an optimal consumption and obtained the characterization of the specific consumption structures. In particular, they showed that the consumption constraint binds up to an endogenously determined stopping time $\tau^* \in [0, T]$, after which it remains slack until T.

For this non-addictive linear habit formation problem in the general incomplete semimartingale markets, we can mimic the path-dependence reduction for the addictive habit formation case, and introduce the auxiliary process $\tilde{c}_t$. However, the non-negative constraint on consumption rate process $c_t \geq 0$ for all $t \in [0, T]$ passes to the path-dependent constraint for $\tilde{c}_t$ as

$$\tilde{c}_t \geq -ze^{\int_0^t (\delta_v - \alpha_v)dv} - \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v)dv} \tilde{c}_s ds, \quad 0 \leq t \leq T,$$

which becomes the first challenge that we need to overcome to employ the convex duality approach, as we know the exponential utility maximization with constraint is generically complicated, especially with path-dependence involved, and some new techniques are needed to be developed. To go a step further, it will be a significant contribution if one can resolve the conjecture of this optimization problem in incomplete markets by showing the constraint on consumption will always cease to bind after an endogenous stopping time. It is exceedingly interesting if one can carry out a similar characterization of the optimal consumption policy in the incomplete market in analogy to the case fully described in the complete market.

Besides of the above technical obstacles, in virtue of the fact that the habit is non-addictive and the utility function is defined on the whole real line, in order to address the extra exogenous random term $\tilde{w}$, we will instead adopt the framework of *Žitković* [77] and extend the auxiliary dual domain for Γ_t to be a set of bounded finitely additive measures on the product space, hence we may need to impose the assumption that stochastic discounting factors are bounded and hence Γ_t is indeed in $\mathbb{L}^1$. This is subject to the future research.

2. The Case of General Nonlinear Habit Formation

Regarding the more general case, possibly nonaddictive, it is a big challenge to provide a positive answer to the utility maximization problem with general nonlinear habit formation $\mathbb{E}[\int_0^T U(t, c_t, Z_t)dt]$, where the instantaneous utility function is strictly increasing in the second argument and strictly deceasing in the third argument. At a first glance, we can not reduce the path dependence of the problem directly as in the linear case. Nevertheless, for each $Y_t \in \mathcal{M}$, we can still partially tackle this problem by resorting to the following Forward-Backward Recursive Stochastic Differential Equations, for which we assume the existence a unique pair of solution (ξ, γ) under some appropriate assumptions,

$$\begin{cases} d\xi_t = \Big(\delta_t I(t, \gamma_t, \xi_t) - \alpha_t \xi_t\Big)dt, \qquad \xi_0 = z, \\ \gamma_t + \delta_t \mathbb{E}\Big[\int_t^T e^{-\int_t^s \alpha_v dv} U_2(t, I(\gamma_s, \xi_s), \xi_s) ds \Big| \mathcal{F}_t\Big] = yY_t, \qquad \gamma_T = y\frac{d\mathbb{Q}}{d\mathbb{P}}. \end{cases}$$

where $I(t, \cdot, z)$ denotes the inverse of $U_1(t, \cdot, z)$. See the proof of the existence of the solutions to the above Forward-Backward Recursive Stochastic Differential Equations in *Detemple and Zapatero* [25] under some Lipschitz and growth conditions.

It is straight forward to justify that for $\eta_t \triangleq U_2(t, I(t, \gamma_t, \xi_t), \xi_t)$,

$$\big\langle c, Y \big\rangle = \big\langle c, \gamma \big\rangle + \big\langle Z, \eta \big\rangle - z\Big(\big\langle 0, \gamma \big\rangle + \big\langle \tilde{w}, \eta \big\rangle\Big).$$

In light of the above equality, we can therefore consider the multivariate utility maximization:

$$\sup_{\bar{c} \in \bar{\mathcal{A}}(x,z)} \mathbb{E}\Big[\int_0^T U(t, \bar{c}_t) dt\Big], \quad \text{where } \bar{c}_t = (c_t, Z_t).$$

where the vector process $\bar{c}$ is taken in the set:

$$\bar{\mathcal{A}}(x,z) = \{(c, Z) \in (\mathbb{L}^0_+)^2 : \big\langle (c, Z), (\gamma, \eta) \big\rangle \leq xy + z\big\langle (0, \tilde{w}), (\gamma, \eta) \big\rangle\}$$

For this, we can naively define the corresponding time separable dual optimization problem

$$\inf_{(\gamma,\eta)} \mathbb{E}\Big[\int_0^T V(t, \gamma_t, \eta_t) dt\Big],$$

with the shadow random endowment vector $(0, \tilde{w})$. However, notice that η_t is a negative process by its definition, so the 2-dimensional Bipolar results with respect to $\bar{\mathcal{A}}(x,z)$ are missing. In addition, we also need to incorporate the state constraint that $Z_t = Z((c_s)_{0 \leq s \leq t}, t)$ to this 2-dimensional optimization problem to address the path dependence of Z_t on c_t. This interesting but complicated open problem is scheduled to be addressed in my future sequel work.

3. Unobservable Hidden Stochastic Process in Discounting Factors

To generalize the result in our second Project with partial observations, one can assume additionally that the discounting factors α_t and δ_t are driven by some unobservable hidden Markov processes which satisfy the mean reverting diffusion SDEs. As we have pointed out in Remark 2.2.1, this assumption makes more sense since the stochastic feature of α_t and δ_t can capture the investor's changes over his habit formation impact due to the random external market influences and other time inconsistent internal psychological effects. It ends up to be a path-dependent stochastic control problem where we can apply multi-dimensional Kalman-Bucy filtering. This problem is generically harder since the corresponding value function can not be decoupled due to the special structure of Habit Formation process, and it will be very exciting if one is capable to show the existence of classical solution to the relatively complicated HJB equation and then prove verification arguments rigorously.

Another possible realistic way to extend our current model is to assume that the investor also receives stochastic income at time t with rate e_t, where the process e_t evolves as

$$de_t = \tilde{\mu}_t e_t dt + \sigma_e e_t dB_t,$$

and it is observable by the investor. However, we can assume similarly that the drift process $\tilde{\mu}_t$ is unobservable to the investor which follows a different mean reverting OU process. Again, this problem fits into the multi-dimensional Kalman-Bucy filtering, and we can perform the similar procedure to solve the HJB equation analytically and justify the rigorous verification arguments.

Appendices

Appendix A

Fully Explicit Solutions of Auxiliary ODEs

Follow the arguments by *Kim and Omberg* [45], we can even solve the auxiliary ODEs (3.3.25), (3.3.26), (3.3.27), (3.3.28) and (3.3.29) fully explicitly depending on the risk aversion constant p and all the market coefficients σ_S, σ_μ, λ, ρ:

A.1 The Normal Solution

The condition for the Normal solution is

$$\Delta \triangleq \lambda^2 - \frac{2\lambda p\rho\sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} > 0, \tag{A.1.1}$$

and then we define:

$$\xi = \sqrt{\Delta} = \sqrt{\gamma_2^2 - \gamma_1\gamma_3}, \qquad \gamma_1 = \frac{(1-p+p\rho^2)}{1-p}\sigma_\mu^2,$$

$$\gamma_2 = -\lambda + \frac{p\rho\sigma_\mu}{(1-p)\sigma_S}, \qquad \gamma_3 = \frac{p}{(1-p)\sigma_S^2},$$

$$\xi_1 = \frac{\sqrt{(1-\rho^2)\sigma_\mu^2 + (\lambda\sigma_S + \rho\sigma_\mu)^2}}{\sigma_S}.$$

We can solve the equations (3.3.25), (3.3.26), (3.3.27), (3.3.28) and (3.3.29) as:

$$
\begin{aligned}
a(t;s) =& \frac{p(1-e^{2\xi(t-s)})}{2(1-p)\sigma_S^2\Big[2\xi-(\xi+\gamma_2)(1-e^{2\xi(t-s)})\Big]},\\
b(t;s) =& \frac{p\lambda\bar{\mu}(1-e^{\xi(t-s)})^2}{2(1-p)\sigma_S^2\xi\Big[2\xi-(\xi+\gamma_2)(1-e^{2\xi(t-s)})\Big]},\\
c(t;s) =& \frac{p}{2(1-p)\sigma_S^2}\Big(\frac{\lambda^2\bar{\mu}^2}{\xi^2}-\frac{\sigma_\mu^2}{\xi+\gamma_2}\Big)(s-t)\\
&+\frac{p\lambda^2\bar{\mu}^2\Big[(\xi+2\gamma_2)e^{2\xi(t-s)}-4\gamma_2 e^{\xi(t-s)}+2\gamma_2-\xi\Big]}{2(1-p)\sigma_S^2\xi^3\Big[2\xi-(\xi+\gamma_2)\Big(1-e^{2\xi(t-s)}\Big)\Big]}\\
&-\frac{p\sigma_\mu^2}{2(1-p)\sigma_S^2(\xi^2-\gamma_2^2)}\log\Big|\frac{2\xi-(\xi+\gamma_2)(1-e^{2\xi(t-s)})}{2\xi}\Big|,
\end{aligned}
$$

$$
\begin{aligned}
f(t;s) =& -\frac{1}{2\sigma_S}\frac{1-e^{2\xi_1(t-s)}}{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}},\\
g(t;s) =& \frac{1}{2}\log\Big(\frac{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1 e^{\xi_1(t-s)}}\Big)\\
&-\frac{(1-p)(1-\rho^2)}{2(1-p+p\rho^2)}\log\Big(\frac{(\sigma_S\xi+\lambda\sigma_S+\frac{\rho\sigma_\mu p}{1-p})+(\sigma_S\xi-\lambda\sigma_S-\frac{\rho\sigma_\mu p}{1-p})e^{2\xi(t-s)}}{2\sigma_S\xi e^{\xi(t-s)}}\Big)\\
&-\frac{\rho^2\lambda(s-t)}{2(1-p+p\rho^2)}-\frac{\rho\sigma_\mu(s-t)}{2(1-p+p\rho^2)\sigma_S},
\end{aligned}
$$

The condition for the bounded Normal solution is

$$\gamma_3>0,\quad \text{or}\quad \gamma_1>0,\quad \text{or}\quad \gamma_2<0. \tag{A.1.2}$$

The condition for the explosive solution and the critical point is

$$\gamma_3<0,\quad \gamma_1<0,\quad \text{and}\quad \gamma_2>0,$$

$$s-t=\frac{1}{2\xi}\log\Big(\frac{\gamma_2+\xi}{\gamma_2-\xi}\Big).$$

Remark A.1.1. *By observation, if $p < 0$, the conditions* (A.1.1) *and* (A.1.2) *hold, and we have $a(t; s) \leq 0$ is a bounded solution as well as $1 - 2a(t; s)\hat{\Omega}(t) > 1 > 0$ and $1 - f(t; s)\hat{\Omega}(t) > 1 > 0$, hence we can finally conclude the solutions of ODEs* (3.3.22), (3.3.23), (3.3.24) *are all bounded on $0 \leq t \leq s \leq T$. We also notice that $A(t) = \frac{a(t)}{(1-p)(1-2a(t)\hat{\Omega}(t))} \leq 0$, on $0 \leq t \leq s \leq T$.*

A.2 The Hyperbolic Solution

The condition for the Hyperbolic solution is

$$\Delta \triangleq \lambda^2 - \frac{2\lambda p\rho\sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} = 0,$$

together with

$$\gamma_2 = -\lambda + \frac{p\rho\sigma_\mu}{(1-p)\sigma_S} \neq 0,$$

Then we can solve (3.3.25), (3.3.26), (3.3.27), (3.3.28) and (3.3.29) as:

$$\begin{aligned}
a(t; s) =& \frac{-1}{2\gamma_1(s - t - \frac{1}{\gamma_2})} - \frac{\gamma_2}{2\gamma_1}, \\
b(t; s) =& -\frac{2\lambda\bar{\mu}}{4\gamma_1\gamma_2(s - t - \frac{1}{\gamma_2})} - \frac{\gamma_2\lambda\bar{\mu}(s - t + \frac{1}{\gamma_2})}{2\gamma_1}, \\
c(t; s) =& \frac{\gamma_2\sigma_\mu^2(s-t)}{2\gamma_1} + \frac{\lambda^2\bar{\mu}^2\gamma_2^2(s - t - \frac{4}{\gamma_2})(s-t)^3}{24\gamma_1(s - t - \frac{1}{\gamma_2})} + \frac{\sigma_\mu^2 \log\left|\frac{1}{2}(s-t)\gamma_2 - 1\right|}{\gamma_1}, \\
f(t; s) =& -\frac{1}{2\sigma_S}\frac{1 - e^{2\xi_1(t-s)}}{(\sigma_S\xi_1 + \lambda\sigma_S + \rho\sigma_\mu) + (\sigma_S\xi_1 - \lambda\sigma_S - \rho\sigma_\mu)e^{2\xi_1(t-s)}}, \\
g(t; s) =& \frac{1}{2}\log\Big(\frac{(\sigma_S\xi_1 + \lambda\sigma_S + \rho\sigma_\mu) + (\sigma_S\xi_1 - \lambda\sigma_S - \rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1 e^{\xi_1(t-s)}}\Big) \\
&- \frac{(\lambda\sigma_S + \rho\sigma_\mu)}{2\sigma_S}(s-t) + \frac{\sigma_\mu^2(1-\rho^2)}{2\gamma_1}\Big[\log\Big|1 + \gamma_2(t-s)\Big| - \gamma_2(s-t)\Big].
\end{aligned}$$

The condition for the bounded Hyperbolic solution is

$$\gamma_2 < 0.$$

The condition for the explosive solution and the critical point is

$$\gamma_2 > 0, \quad \text{and} \quad s - t = \frac{1}{\gamma_2}.$$

A.3 The Polynomial Solution

The condition for the Polynomial solution is

$$\Delta \triangleq \lambda^2 - \frac{2\lambda p \rho \sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} = 0,$$

together with

$$\gamma_2 = -\lambda + \frac{p\rho\sigma_\mu}{(1-p)\sigma_S} = 0,$$

Then we can solve (3.3.25), (3.3.26), (3.3.27), (3.3.28) and (3.3.29) as:

$$\begin{aligned}
a(t;s) =& \frac{p}{2(1-p)\sigma_S^2}(s-t), \\
b(t;s) =& \frac{p}{2(1-p)\sigma_S^2}\lambda\bar{\mu}(s-t)^2, \\
c(t;s) =& -\frac{p}{4(1-p)\sigma_S^2}\sigma_\mu^2(s-t)^2 + \frac{p}{6(1-p)\sigma_S^2}\lambda^2\bar{\mu}^2(s-t)^3,
\end{aligned}$$

$$\begin{aligned}
f(t;s) =& -\frac{1}{2\sigma_S}\frac{1-e^{2\xi_1(t-s)}}{(\sigma_S\xi_1 + \lambda\sigma_S + \rho\sigma_\mu) + (\sigma_S\xi_1 - \lambda\sigma_S - \rho\sigma_\mu)e^{2\xi_1(t-s)}}, \\
g(t;s) =& \frac{1}{2}\log\left(\frac{(\sigma_S\xi_1 + \lambda\sigma_S + \rho\sigma_\mu) + (\sigma_S\xi_1 - \lambda\sigma_S - \rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1 e^{\xi_1(t-s)}}\right) \\
& - \frac{(\lambda\sigma_S + \rho\sigma_\mu)}{2\sigma_S}(s-t) - \frac{\sigma_\mu^2(1-\rho^2)p}{4(1-p)\sigma_S^2}(s-t)^2.
\end{aligned}$$

All Polynomial solutions are bounded.

A.4 The Tangent Solution

The condition for the Tangent solution is

$$\Delta \triangleq \lambda^2 - \frac{2\lambda p\rho\sigma_\mu}{(1-p)\sigma_S^2} - \frac{p\sigma_\mu^2}{(1-p)\sigma_S^2} < 0,$$

Now, we define

$$\zeta = \sqrt{-\Delta}, \quad \varpi = \tan^{-1}\left(\frac{\gamma_2}{\zeta}\right),$$

Then we can solve (3.3.25), (3.3.26), (3.3.27), (3.3.28) and (3.3.29) as:

$$\begin{aligned}
a(t;s) =& \frac{\zeta}{2\gamma_1}\tan\Big(\zeta(s-t)+\varpi\Big) - \frac{\gamma_2}{2\gamma_1},\\
b(t;s) =& \frac{\lambda\bar{\mu}}{\gamma_1}\Big[-1-\tan(\varpi)\tan(\zeta(s-t)+\varpi)+\sec(\varpi)\sec(\zeta(s-t)+\varpi)\Big],\\
c(t;s) =& \frac{2\lambda^2\bar{\mu}^2\gamma_2\sqrt{\gamma_2^2+\zeta^2}}{2\gamma_1\zeta}\Big[\sec(\varpi)-\sec(\zeta(s-t)+\varpi)\Big]\\
&+\frac{\lambda^2\bar{\mu}^2(2\gamma_2+\zeta^2)}{2\gamma_1\zeta^3}\Big[\tan(\zeta(s-t)+\varpi)-\tan(\varpi)\Big]\\
&-\frac{\lambda^2\bar{\mu}^2(\gamma_2^2+\zeta^2)-\gamma_2\zeta^2\sigma_\mu^2}{2\gamma_1\zeta^2}+\frac{\sigma_\mu^2}{2\gamma_1}\log\Big(\sec(\varpi)\cos(\zeta(s-t)+\varpi)\Big),
\end{aligned}$$

$$\begin{aligned}
f(t;s) =& -\frac{1}{2\sigma_S}\frac{1-e^{2\xi_1(t-s)}}{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}},\\
g(t;s) =& \frac{1}{2}\log\Big(\frac{(\sigma_S\xi_1+\lambda\sigma_S+\rho\sigma_\mu)+(\sigma_S\xi_1-\lambda\sigma_S-\rho\sigma_\mu)e^{2\xi_1(t-s)}}{2\sigma_S\xi_1 e^{\xi_1(t-s)}}\Big)\\
&-\frac{(\lambda\sigma_S+\rho\sigma_\mu)}{2\sigma_S}(s-t)-\sigma_\mu^2(1-\rho^2)\Big[\frac{1}{2\gamma_1}\log\Big|\frac{\cos(\zeta(t-s)+\varpi)}{\cos(\varpi)}\Big|\\
&-\frac{\gamma_2}{2\gamma_1}(s-t)\Big].
\end{aligned}$$

All Tangent solutions are explosive solutions and the critical point is

$$s-t=\frac{\pi}{2\zeta}-\frac{1}{\zeta}\tan^{-1}(\frac{\gamma_2}{\zeta}).$$

Appendix B

Proof of Kalman Bucy Filtering Theorem

In order to prove the Kalman-Bucy filtering theorem, we need the following propositions and lemmas as preparations.

Proposition B.0.1. *The Innovation Process defined by* $\hat{W}_t \triangleq W_t + \int_0^t \frac{D(s)}{E(s)}\Big(R_s - \hat{R}_s\Big)ds$ *is an* $\hat{\mathfrak{F}}_t$*-adapted Brownian motion.*

Proof. First, we need to show that $\hat{W}_t$ is $\hat{\mathfrak{F}}_t$ adapted. This follows readily by observing

$$\hat{W}_t = W_t + \int_0^t \frac{D(s)}{E(s)} R_s ds - \int_0^t \frac{D(s)}{E(s)} \hat{R}_s ds$$

and the fact that $W_t + \int_0^t \frac{D(s)}{E(s)} R_s ds = \int_0^t \frac{1}{E(s)} dH_s$ and $\hat{R}_t$ are both $\hat{\mathfrak{F}}_t = \sigma(H_s :\ 0 \leq s \leq t)$ adapted.

With $0 \leq s \leq t \leq T$, by Tower property, we have:

$$\begin{aligned}
\mathbb{E}\Big[\hat{W}_t \Big| \hat{F}_s\Big] - \hat{W}_s &= \mathbb{E}\Big[\int_s^t \frac{D(u)}{E(u)}\Big(R_u - \hat{R}_u\Big)du + W_t - W_s \Big| \hat{\mathfrak{F}}_s\Big] \\
&= \mathbb{E}\Big[\int_s^t \frac{D(u)}{E(u)}\Big(\mathbb{E}\Big[R_u \Big| \hat{\mathfrak{F}}_u\Big] - \hat{R}_u\Big)du \Big| \hat{\mathfrak{F}}_s\Big] \\
&+ \mathbb{E}\Big[\mathbb{E}\Big[W_t - W_s \Big| \mathfrak{F}_s\Big] \Big| \hat{\mathfrak{F}}_s\Big] = 0.
\end{aligned}$$

Hence, $\hat{W}_t$ is a continuous $\hat{\mathfrak{F}}_t$ adapted martingale with quadratic variation $\langle \hat{W} \rangle_t = \langle W \rangle_t = t$, and Levy's characterization lemma implies $\hat{W}$ is an $(\hat{\mathfrak{F}}_t)_{0 \leq t \leq T}$ adapted Brownian motion. □

Theorem B.0.2. *Every $\hat{\mathbb{F}}$-local martingale M admits a representation of the form:*

$$M_t = M_0 + \int_0^t \phi_s d\hat{W}_s,$$

where ϕ_t is $\hat{\mathfrak{F}}_t$ adapted and $\int_0^T \phi_t^2 dt < \infty$, a.s.. If M happens to be a square integrable martingale, then ϕ_t can be chosen so that $\mathbb{E}\Big[\int_0^T \phi_t^2 dt\Big] < \infty$.

Proof. For fixed $n \in \mathbb{N}$, we define the stopping time

$$\tau_n = \inf\Big\{t \geq 0 : \Big|\int_0^t \frac{D(s)}{E(s)}\hat{R}_s d\hat{W}_s\Big| + \int_0^t \Big(\frac{D(s)}{E(s)}\hat{R}_s\Big)^2 ds \leq n\Big\} \wedge T.$$

We denote Λ_t the exponential local martingale as

$$\Lambda_t \triangleq \exp\Big(-\int_0^t \Big(\frac{D(s)}{E(s)}\hat{R}_s\Big) d\hat{W}_s - \frac{1}{2}\int_0^t \Big(\frac{D(s)}{E(s)}\hat{R}_s\Big)^2 ds\Big), \quad 0 \leq t \leq T \quad \text{(B.0.1)}$$

then the stopped process $\Lambda_t^n \triangleq \Lambda_{t\wedge\tau_n}$ is a $(\Omega, (\hat{\mathfrak{F}}_{t\wedge\tau_n})_{0\leq t\leq T}, \mathbb{P})$ UI-martingale. Since $\hat{W}_{t\wedge\tau_n}$ is a Brownian motion under $(\Omega, (\hat{\mathfrak{F}}_{t\wedge\tau_n})_{0\leq t\leq T}, \mathbb{P})$, Girsanov Theorem deduces the process:

$$L_t^n = \int_0^{t\wedge\tau_n} \frac{1}{E(s)} dH_s = \hat{W}_{t\wedge\tau_n} + \int_0^{t\wedge\tau_n} \Big(\frac{D(s)}{E(s)}\hat{R}_s\Big) ds$$

is a Brownian motion under $(\Omega, (\hat{\mathfrak{F}}_{t\wedge\tau_n})_{0\leq t\leq T}, \mathbb{P}^n)$, where the probability measure $\mathbb{P}^n$ is defined via

$$\frac{d\mathbb{P}^n}{d\mathbb{P}} = \Lambda_T^n.$$

For the future purpose, we also define the process

$$\begin{aligned}\Theta_t^n = \frac{1}{\Lambda_t^n} &= \exp\Big(\int_0^{t\wedge\tau_n} \Big(\frac{D(s)}{E(s)}\hat{R}_s\Big) d\hat{W}_s + \frac{1}{2}\int_0^{t\wedge\tau_n} \Big(\frac{D(s)}{E(s)}\hat{R}_s\Big)^2 ds\Big) \\ &= \exp\Big(\int_0^{t\wedge\tau_n} \Big(\frac{D(s)}{E(s)}\hat{R}_s\Big) dL_s^n - \frac{1}{2}\int_0^{t\wedge\tau_n} \Big(\frac{D(s)}{E(s)}\hat{R}_s\Big)^2 ds\Big).\end{aligned}$$

Hence we will have:

$$\Lambda_t^n = 1 - \int_0^{t\wedge\tau_n} \Lambda_s^n\Big(\frac{D(s)}{E(s)}\hat{R}_s\Big)d\hat{W}_s, \quad \Theta_t^n = 1 + \int_0^{t\wedge\tau_n} \Theta_s^n\Big(\frac{D(s)}{E(s)}\hat{R}_s\Big)dL_s^n.$$

Notice $\hat{\mathfrak{F}}_{t\wedge\tau_n} = \mathfrak{F}_{t\wedge\tau_n}^{L^n}$ is the natural filtration generated by the Brownian motion L_t^n under the probability measure $\mathbb{P}^n$, we have for any $(\mathbb{P}, \hat{\mathfrak{F}}_{t\wedge\tau_n})$-local martingale $M_{t\wedge\tau_n}$, with its localizing sequence $(\tilde{\tau}_k)_{k=1}^{\infty}$, where $\tilde{\tau}_k = \inf\{t : |M_t| = k\}$, we will have $(M_{t\wedge\tau_n\wedge\tilde{\tau}_k})$ is a $(\mathbb{P}, \hat{\mathfrak{F}}_{t\wedge\tau_n})$-martingale. Hence, by *Bayes's rule*:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^n}\Big[\Theta_t^n M_{t\wedge\tau_n\wedge\tilde{\tau}_k}\Big|\hat{\mathfrak{F}}_{s\wedge\tau_n}\Big] =& \frac{1}{\Lambda_s^n}\mathbb{E}\Big[\Lambda_t^n\Theta_t^n M_{t\wedge\tau_n\wedge\tilde{\tau}_k}\Big|\hat{\mathfrak{F}}_{s\wedge\tau_n}\Big]\\ =&\Theta_s^n\mathbb{E}\Big[M_{t\wedge\tau_n\wedge\tilde{\tau}_k}\Big|\hat{\mathfrak{F}}_{s\wedge\tau_n}\Big] = \Theta_s^n M_{s\wedge\tau_n\wedge\tilde{\tau}_k}.\end{aligned}$$

So that $(\Theta_t^n M_{t\wedge\tau_n})_{0\le t\le T}$ is a $(\mathbb{P}^n, \hat{\mathfrak{F}}_{t\wedge\tau_n})$-local martingale, and consequently, it has a stochastic integral representation with respect to the Brownian motion L_t^n such that for $0 \le t \le T$,

$$\Theta_t^n M_{t\wedge\tau_n} = \Theta_0^n M_0 + \int_0^{t\wedge\tau_n} \Psi_s dL_s^n = \Theta_0^n M_0 + \int_0^{t\wedge\tau_n} \Psi_s\Big(d\hat{W}_s + \frac{D(s)}{E(s)}\hat{R}_s ds\Big),$$

for some $\hat{\mathfrak{F}}_{t\wedge\tau_n}$ adapted process Ψ_t satisfying $\int_0^T \Psi_t^2 dt < +\infty, \quad a.s.$.

By applying Integration by parts formula, we finally have:

$$\begin{aligned}M_{t\wedge\tau_n} =& \Big(\Theta_t^n M_{t\wedge\tau_n}\Big)\Lambda_t^n\\ =&M_0 + \int_0^{t\wedge\tau_n}\Big(\Theta_s^n M_s\Big)d\Lambda_s^n + \int_0^{t\wedge\tau_n}\Lambda_s^n d\Big(\Theta_s^n M_s\Big) + \big\langle\Theta^n M, \Lambda^n\big\rangle_{t\wedge\tau_n}\end{aligned}$$

which then gives the equalities

$$\begin{aligned}
M_{t\wedge\tau_n} =& M_0 + \int_0^{t\wedge\tau_n} \Theta_s^n M_s\Big(- \Lambda_s^n \frac{D(s)}{E(s)} \hat{R}_s d\hat{W}_s\Big) \\
& + \int_0^{t\wedge\tau_n} \Lambda_s^n \Psi_s\Big(d\hat{W}_s + \frac{D(s)}{E(s)}\hat{R}_s ds\Big) - \int_0^{t\wedge\tau_n} \Lambda_s^n \frac{D(s)}{E(s)}\hat{R}_s\Psi_s d\big\langle \hat{W}\big\rangle_s \\
=& M_0 + \int_0^{t\wedge\tau_n} \Big(\Lambda_s^n\Psi_s - M_s\frac{D(s)}{E(s)}\hat{R}_s\Big)d\hat{W}_s \\
=& M_0 + \int_0^{t\wedge\tau_n} \Big(\Lambda_s\Psi_s - M_s\frac{D(s)}{E(s)}\hat{R}_s\Big)d\hat{W}_s
\end{aligned}$$

from which, we get the stochastic integral representation with respect to $\hat{W}_t$ for each $n \in \mathbb{N}$, and by letting $n \longrightarrow +\infty$, we get the conclusion. $\square$

Lemma B.0.3. *Consider two $\mathbb{F}$-adapted process $P,\ K$ with $\mathbb{E}[|P_t|] < +\infty,\ \forall\, t \in [0,T]$ and $\mathbb{E}[\int_0^T |K_t|dt] < +\infty$. If $J_t = P_t - \int_0^T K_s ds$ is an $\mathbb{F}$-martingale, then*

$$\hat{J}_t = \hat{P}_t - \int_0^T \hat{K}_s ds \tag{B.0.2}$$

is an $\hat{\mathbb{F}}$-martingale.

Proof. For $s \le t$, by writing $\int_0^t K_u du = \int_0^s K_u du + \int_s^t K_u du$ and using the fact that J is an $\mathbb{F}$ martingale, we have

$$\mathbb{E}\Big[P_t - \int_0^s K_u du - \int_s^t K_u du \Big| \mathcal{F}_s\Big] = P_s - \int_0^s K_u du.$$

Hence, $\mathbb{E}\Big[P_t - \int_s^t K_u du\Big|\mathcal{F}_s\Big] = P_s$.

Now, we can show the following equalities

$$
\begin{aligned}
\mathbb{E}\Big[\hat{J}_t\Big|\hat{\mathfrak{F}}_s\Big] &= \mathbb{E}\Big[\hat{P}_t - \int_0^s \hat{K}_u du - \int_s^t \hat{K}_u du\Big|\hat{\mathfrak{F}}_s\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[P_t\Big|\hat{\mathfrak{F}}_t\Big] - \int_s^t \mathbb{E}\Big[K_u\Big|\hat{\mathfrak{F}}_u\Big]du\Big|\hat{\mathfrak{F}}_s\Big] - \int_0^s \hat{K}_u du \\
&= \mathbb{E}\Big[P_t\Big|\hat{\mathfrak{F}}_s\Big] - \int_s^t \mathbb{E}\Big[K_u\Big|\hat{\mathfrak{F}}_s\Big]du - \int_0^s \hat{K}_u du \\
&= \mathbb{E}\Big[\mathbb{E}\Big[P_t\Big|\mathfrak{F}_s\Big]\Big|\hat{\mathfrak{F}}_s\Big] - \int_s^t \mathbb{E}\Big[\mathbb{E}\Big[K_u\Big|\mathfrak{F}_s\Big]\Big|\hat{\mathfrak{F}}_s\Big]du - \int_0^s \hat{K}_u du \\
&= \mathbb{E}\Big[\mathbb{E}\Big[P_t - \int_s^t K_u du\Big|\mathfrak{F}_s\Big]\Big|\hat{\mathfrak{F}}_s\Big] - \int_0^s \hat{K}_u du \\
&= \mathbb{E}\Big[P_s\Big|\hat{\mathfrak{F}}_s\Big] - \int_0^s \hat{K}_u du = \hat{P}_s - \int_0^s \hat{K}_u du = \hat{J}_s.
\end{aligned}
$$

□

Proof. *(Kamlman-Bucy Filtering theorem)*:

Observe the dynamic of process R_t and H_t, it's easy to verify that $\mathbb{E}\Big[\int_0^T R_t^2 dt\Big] < +\infty$ and $\mathbb{E}\Big[\int_0^T H_t^2 dt\Big] < +\infty$.

Meanwhile, we have $\int_0^t C(s)dB_s$ is an $\mathbb{F}$-martingale, hence by Theorem $B.0.2$ and Lemma $B.0.3$, $(\widehat{\int_0^t C(s)dB_s})$ is an $\hat{\mathbb{F}}$-martingale, and has a representation as $(\widehat{\int_0^t C(s)dB_s}) = \int_0^t \phi_s d\hat{W}_s$, for an $\hat{\mathbb{F}}$-adapted process ϕ_t such that $\mathbb{E}\Big[\int_0^T \phi_t^2 dt\Big] < +\infty$.

Rewrite the SDE for R_t, we will arrive at

$$d\hat{R}_t = A(t)\hat{R}_t dt + \phi_t d\hat{W}_t, \tag{B.0.3}$$

One the other hand, the definition of Innovation Process implies

$$dH_t = D(t)\hat{R}_t dt + E(t)d\hat{W}_t. \tag{B.0.4}$$

Now, we intend to apply integration by parts formula for R_tH_t in two different ways: On the one hand, by the linear SDE (1.3.1), we deduce that

$$\begin{aligned}R_tH_t =& R_0H_0 + \int_0^t R_s dH_s + \int_0^t H_s dR_s + \big\langle H, R\big\rangle_t \\ =& R_0H_0 + \int_0^t \Big[D(s)R_s^2 + A(s)H_sR_s + C(s)E(s)\rho\Big]ds \\ &+ \int_0^t E(s)R_s dW_s + \int_0^t C(s)H_s dB_s.\end{aligned}$$

And by assumption that $\mathbb{E}\Big[\int_0^T R_t^2 dt\Big] < +\infty$ and $\mathbb{E}\Big[\int_0^T H_t^2 dt\Big] < +\infty$, we have $\int_0^t E(s)R_s dW_s + \int_0^t C(s)H_s dB_s$ is an $\mathbb{F}$-martingale. Hence, by Lemma $B.0.3$ again, we get

$$\widehat{R_tH_t} = \hat{R}_tH_t = \int_0^t \Big[D(s)\widehat{R_s^2} + A(s)H_s\hat{R}_s + C(s)E(s)\Big]ds + \hat{\mathbb{F}}\text{-martingale.} \tag{B.0.5}$$

On the other hand, from $(B.0.3)$ and $(B.0.4)$, we can also derive that

$$\begin{aligned}\hat{R}_tH_t &= \hat{R}_0H_0 + \int_0^t \hat{R}_s dH_s + \int_0^t H_s d\hat{R}_s + \langle H, \hat{R}\rangle_t \\ &= R_0H_0 + \int_0^t \Big[D(s)\hat{R}_s^2 + A(s)H_s\hat{R}_s + E(s)\phi_s\Big]ds \\ &+ \int_0^t \Big[E(s)\hat{R}_s + H_s\phi_s\Big]d\hat{W}_s \\ &= R_0H_0 + \int_0^t \Big[D(s)\hat{R}_s^2 + A(s)H_s\hat{R}_s + E(s)\phi_s\Big]ds + \ \hat{\mathbb{F}}\text{-martingale.}\end{aligned} \tag{B.0.6}$$

Comparing $(B.0.5)$ and $(B.0.6)$, the difference between the bounded variation parts is a continuous $\hat{\mathbb{F}}$-martingale, hence a constant, and is identically zero. Therefore, we can get the following dynamics for the process

$\hat{R}_t$

$$d\hat{R}_t = A(t)\hat{R}_t dt + \Big[\frac{D(t)}{E(t)}\hat{\Omega}_t + C(t)\Big]d\hat{W}_t, \tag{B.0.7}$$

with $\hat{R}_0 = \mu$.

Similarly, we do the same computation by using the integration by parts formula to $\widehat{R_t^2}H_t$, and by Itô's Lemma, we can write down

$$d\widehat{R_t^2} = \Big(C^2(t) + 2A(t)\widehat{R_t^2}\Big)dt + \Big[\frac{D(t)}{E(t)}(\widehat{R_t^3} - \hat{R}_t\widehat{R_t^2}) + 2\rho C(t)\hat{R}_t\Big]d\hat{W}_t, \tag{B.0.8}$$

with $\widehat{R_0^2} = \mu^2 + \theta$.

But for a random variable $X \sim N(m, s^2)$ with Normal distribution, we have

$$\mathbb{E}[X^3] = m(m^2 + 3s^2),$$

hence, we know that

$$\widehat{R_t^3} = \mathbb{E}\Big[R_t^3\Big|\hat{\mathfrak{F}}_t\Big] = \mathbb{E}\Big[R_t\Big|\hat{\mathfrak{F}}_t\Big]\bigg(\Big(\mathbb{E}\Big[R_t\Big|\hat{\mathfrak{F}}_t\Big]\Big)^2 + 3\,Var\Big[R_t\Big|\hat{\mathfrak{F}}_t\Big]\bigg) = \hat{R}_t\Big[(\hat{R}_t)^2 + 3\hat{\Omega}_t\Big],$$

and therefore

$$\widehat{R_t^3} - \hat{R}_t\widehat{R_t^2} = \hat{R}_t\Big[(\hat{R}_t)^2 + 3\hat{\Omega}_t - \widehat{R_t^2}\Big] = 2\hat{\Omega}_t\hat{R}_t.$$

Using this fact, and $(B.0.8)$, $(B.0.7)$, we can derive that

$$\begin{aligned} d\hat{\Omega}_t &= d\Big[\widehat{R_t^2} - (\hat{R}_t)^2\Big] \\ &= \Big(C^2(t) + 2A(t)\widehat{R_t^2}\Big)dt + \Big[\frac{D(t)}{C(t)}(2\hat{\Omega}_t\hat{R}_t) + 2\rho C(t)\hat{R}_t\Big]d\hat{W}_t \\ &\quad - 2\hat{R}_t d\hat{R}_t - d\big\langle\hat{R}\big\rangle_t, \end{aligned}$$

which simplifies to the deterministic Riccati equation:

$$\frac{d\hat{\Omega}_t}{dt} = -\frac{D^2(t)}{E^2(t)}\hat{\Omega}_t^2 + 2\Big[A(t) - \frac{C(t)D(t)\rho}{E(t)}\Big]\hat{\Omega}_t + C^2(t)(1-\rho^2)$$

with initial condition $\hat{\Omega}_0 = \theta$. And the proof is complete. □

Appendix C

The Reasonable Asymptotic Elasticity of Utility Functions

We show the proof of Lemma 2.2.2 and Corollary 2.2.3 concerning the Reasonable Asymptotic Elasticity Conditions for completeness of this dissertation.

Proof of Lemma 2.2.2. It follows from the definition of the Reasonable Asymptotic Elasticity that $AE[U]_\infty$ equals the infimum over all γ such that (ii) holds true. We shall show that for each of the above four conditions the inf of the γ for which they hold true is the same.

$(i) \Leftrightarrow (ii)$ To show that $(ii) \Rightarrow (i)$, fix $x > 0, \gamma > 0$ and compare the two functions

$$F(t,\lambda) = U(t,\lambda x) \quad \text{and} \quad G(t,\lambda) = \lambda^\gamma U(t,x), \quad \lambda > 1, \quad t \in [0,T].$$

Here F and G are differentiable with respect to λ, $F(t,1) = G(t,1)$ for all $t \in [0,T]$, and if (ii) holds true then, for $x > x_0$ and any $t \in [0,T]$

$$F'(t,1) = xU'(t,x) < \gamma U(t,x) = G'(t,1),$$

hence we have $F(t,\lambda) < G(t,\lambda)$ for $\lambda \in (1, 1+\epsilon)$ and all $t \in [0,T]$, for some $\epsilon > 0$. To show that $F(t,\lambda) < G(t,\lambda)$ for all $\lambda > 1$ and $t \in [0,T]$, let

$\hat{\lambda}(t) = \inf\{\lambda > 1 : F(t,\lambda) = G(t,\lambda)\}$ and suppose that $\hat{\lambda}(t) < \infty$ for some $t \in [0,T]$. Note that we must have $F'(t,\hat{\lambda}(t)) \geq G'(t,\hat{\lambda}(t))$, which leads to a contradiction as it follows from (ii) that

$$\begin{aligned} F'(t,\hat{\lambda}(t)) &= x\,U'(t,\hat{\lambda}(t)x) < \frac{\gamma}{\hat{\lambda(t)}}\,U(t,\hat{\lambda}(t)x) = \frac{\gamma}{\hat{\lambda}(t)}F(t,\hat{\lambda}(t)) \\ &= \frac{\gamma}{\hat{\lambda}(t)}G(\hat{\lambda}(t)) = G'(t,\hat{\lambda}(t)). \end{aligned}$$

The reverse implication $(i) \Rightarrow (ii)$ follows from

$$U'(t,x) = \frac{F'(t,1)}{x} \leq \frac{G'(t,1)}{x} = \gamma\frac{U(t,x)}{x}.$$

$(ii) \Leftrightarrow (iv)$ Assuming (ii) we may estimate, for $y < y_0 \triangleq \inf_{t\in[0,T]} U'(t,x_0)$,

$$\begin{aligned} V(t,y) &= \sup_x \Big[U(t,x) - xy\Big] \\ &= U(t,-V'(t,y)) + y\,V'(t,y) \\ &> \frac{1}{\gamma}(-V'(t,y))\,U'(-V'(t,y)) + y\,V'(t,y) = \frac{1-\gamma}{\gamma}y(-V'(t,y)), \end{aligned}$$

which is precisely (iv). Conversely, assuming (iv) we get, for $x > x_0 \triangleq -\inf_{t\in[0,T]} V'(t,y_0)$,

$$\begin{aligned} U(t,x) &= \inf_{y>0}\Big[V(t,y) + xy\Big] \\ &= V(t,U'(t,x)) + x\,U'(t,x) \\ &> \frac{1-\gamma}{\gamma}U'(t,x)(-V'(t,U'(t,x))) + x\,U'(t,x) = \frac{1}{\gamma}x\,U'(t,x), \end{aligned}$$

which is precisely (ii).

$(iii) \Leftrightarrow (iv)$ Just as in the proof of $(i) \Leftrightarrow (ii)$ we compare, for $0 < y \leq y_0 \triangleq \inf_{t\in[0,T]} U'(t,x_0)$ fixed, the functions

$$F(t,\mu) = V(t,\mu y) \quad \text{and} \quad G(t,\mu) = \mu^{\frac{-\gamma}{1-\gamma}}\,V(t,y), \quad 0 < \mu < 1, \quad t \in [0,T]$$

to obtain that (iv) is equivalent to $F(t,\mu) < G(t,\mu)$, for $0 < y \le y_0$ and $0 < \mu < 1$ for all $t \in [0,T]$. This easily implies the equivalence of (iii) and (iv). $\square$

Proof of Corollary 2.2.3. It is enough to show the equivalence between (2.2.10) and (2.2.13), as for the proof, we just follow the exact arguments in the proof of Lemma 2.2.2 by taking function $U(t,x) = -V(t,x)$ for $x > 0$.

Suppose $AE[U]_0 < \infty$ holds true, then there exists a constant $0 < M < \infty$ and $x_0 > 0$ for all $t \in [0,T]$ such that

$$-U'(t,x) > M\frac{U(t,x)}{x}, \quad \text{for } 0 < x \le x_0.$$

Then we can set $\gamma = \frac{M}{1+M}$, we obtain (iv) of Corollary 2.2.3, as we have $M < \infty$, it is clear that $\gamma < 1$, and hence we obtain $AE[V]_\infty < 1$ under the Assumption (2.2.12).

On the other hand, if we assume $AE[V]_\infty < 1$, then (iv) of Corollary 2.2.3 holds, where the infimum of γ is strictly less than 1, i.e. there exists $\gamma_0 < 1$. By setting $M = \frac{\gamma_0}{1-\gamma_0}$, we obtain the existence of constant $M < \infty$, such that

$$-U'(t,x) > M\frac{U(t,x)}{x}, \quad \text{for } 0 < x \le x_0.$$

which is equivalent to the claim that $AE[U]_0 < \infty$. $\square$

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